BOOK I

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CHAPTER IV

MOTION OF A SYSTEM OF BODIES

144. IT is known by observation, that the relative motions of a system of bodies, are entirely independent of any motion common to the whole; hence it is impossible to judge from appearances alone, of the absolute motions of a system of bodies of which we form a part; the knowledge of the true system of the world was retarded, from the difficulty of comprehending the relative motions of projectiles on the earth, which has the double motion of rotation and revolution. But all the motions of the solar system, determined according to this law, are verified by observation.

By article 117, the equation of the motion of a body only differs from that of a particle, by the mass; hence, if only one body be considered, of which m is the mass, the motion of its centre of gravity will be determined from equation (6), which in this case becomes

$$m\left\{\mathrm{X}-\frac{d^2x}{dt^2}\right\}dx+m\left\{\mathrm{Y}-\frac{d^2y}{dt^2}\right\}dy+m\left\{\mathrm{Z}-\frac{d^2z}{dt^2}\right\}dz=0\,.$$

A similar equation may be found for each body in the system, and one condition to be fulfilled is, that the sum of all such equations must be zero;–hence the general equation of a system of bodies is

$$0=\Sigma\, m\left\{\mathrm{X}-\frac{d^2x}{dt^2}\right\}dx+\Sigma\, m\left\{\mathrm{Y}-\frac{d^2y}{dt^2}\right\}dy+\Sigma\, m\left\{\mathrm{Z}-\frac{d^2z}{dt^2}\right\}dz\,, \qquad \textbf{(21)}$$

in which

$$\Sigma\; m\;\mathrm{F}\;d\;s\;=\;0$$

are the sums of the products of each mass by its corresponding component force, for

$$\Sigma\, m\mathrm{X}=m\mathrm{X}+m'\mathrm{X}'+m''\mathrm{X}''+\&c.;$$

and so for the other two. Also

$$\Sigma\, m\frac{d^2x}{dt^2},\quad \Sigma\, m\frac{d^2y}{dt^2},\quad \Sigma\, m\frac{d^2z}{dt^2},$$

are the sums of the products of each mass, by the second increments of the space respectively described by them, in an element of time in the direction of each axis, since

$$\sum m\frac{d^2x}{dt^2}=m\frac{d^2x}{dt^2}+m'\frac{d^2x'}{dt^2}+\&c.$$

the expressions

$$\sum m\frac{d^2y}{dt^2},\quad \sum m\frac{d^2z}{dt^2}$$

have a similar signification.

From this equation all the motions of the solar system are directly obtained.

145. If the forces be invariably supposed to have the same intensity at equal distances from the points to which they are directed, and to vary in some ratio of that distance, all the principles of motion that have been derived from the general equation (6), may be obtained from this, provided the sum of the masses be employed instead of the particle.

146. For example, if the equation, in article 74, be multiplied by $\sum m$ its finite value is found to be[1]

$$\sum m\mathrm{V}^2=\mathrm{C}+2\sum\int m\left(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz\right).$$

This is the Living Force or Impetus of a system, which is the sum of the masses into the square of their respective velocities, and is analogous to the equation[2]

$$\mathrm{V}^2=\mathrm{C}+2v^2,$$

relating to a particle.

147. When the motion of the system changes by insensible degrees, and is subject to the action of accelerating forces, the sum of the indefinitely small increments of the impetus is the same, whatever be the path of the bodies, provided that the points of departure and arrival be the same.

148. When there is a primitive impulse without accelerating forces, the impetus is constant.

149. Impetus is the true measure of labour; for if a weight be raised ten feet, it will require four times the labour to raise an equal weight forty feet. If both these weights be allowed to descend freely by their gravitation, at the end of their fall their velocities will be as 1 to 2; that is, as the square roots of their heights. But the effects produced will be as their masses into the heights from whence they fell, or as their masses into 1 and 4; but these are the squares of the velocities, hence the impetus is the mass into the square of the velocity. Thus the impetus is the true measure of the labour employed to raise the weights, and of the effects of their descent, and is entirely independent of time.

150. The principle of least action for a particle was shown, in article 80, to be expressed by $d\int vds = 0$, when the extreme points of its path are fixed; hence, for a system of bodies, it is

$$\Sigma d\int mvds = 0, \text{ or } \Sigma d\int mv^2dt = 0.$$

Thus the sum of the living forces of a system of bodies is a minimum, during the time that it takes to pass from one position to another.

If the bodies be not urged by accelerating forces, the impetus of the system during a given time, is proportional to that time, therefore the system moves from one given position to another, in the shortest time possible: which is the principle of least action in a system of bodies.

On the Motion of the Centre of Gravity of a System of Bodies

151. In a system of bodies the common centre of gravity of the whole either remains at rest or moves uniformly in a straight line, as if all the bodies of the system were united in that point, and the concentrated forces of the system applied to it.

Demonstration. These properties are derived from the general equation (21) by considering that, if the centre of gravity of the system be moved, each body will have a corresponding and equal motion independent of any motions the bodies may have among themselves: hence each of the virtual velocities $d\,x$, $d\,y$, $d\,z$, will be increased by the virtual velocity of the centre of gravity resolved in the direction of the axes; so that they become

$$d\,x + d\,\overline{x},\ d\,y + d\,\overline{y},\ d\,z + d\,\overline{z}:$$

thus the equation of the motion of a system of bodies is increased by the term,[3]

$$\Sigma.m\left\{\mathrm{X} - \frac{d^2x}{dt^2}\right\}d\,\overline{x} + \Sigma.m\left\{\mathrm{Y} - \frac{d^2y}{dt^2}\right\}d\,\overline{y} + \Sigma.m\left\{\mathrm{Z} - \frac{d^2z}{dt^2}\right\}d\,\overline{z}$$

arising from the consideration of the centre of gravity. If the system be free and unconnected with bodies foreign to it, the virtual velocity of the centre of gravity, is independent of the connexion of the bodies of the system with each other; therefore $d\,\overline{x}$, $d\,\overline{y}$, $d\,\overline{z}$ may each be zero, whatever the virtual velocity of the bodies themselves may be; hence

$$\Sigma.m\left\{\mathrm{X} - \frac{d^2x}{dt^2}\right\} = 0,\quad \Sigma.m\left\{\mathrm{Y} - \frac{d^2y}{dt^2}\right\} = 0,\quad \Sigma.m\left\{\mathrm{Z} - \frac{d^2z}{dt^2}\right\} = 0.$$

But it has been shewn[4] that the co-ordinates of the centre of gravity are,

$$\overline{x} = \frac{\Sigma.mx}{\Sigma.m};\quad \overline{y} = \frac{\Sigma.my}{\Sigma.m};\quad \overline{z} = \frac{\Sigma.mz}{\Sigma.m}.$$

Consequently,

$$d^2\overline{x} = \frac{\sum .md^2x}{\sum .m}; \quad d^2\overline{y} = \frac{\sum .md^2y}{\sum .m}; \quad d^2\overline{z} = \frac{\sum .md^2z}{\sum .m}.$$

Now[5]

$$\sum .md^2x = dt^2 . \sum .m\text{X}; \quad \sum .md^2y = dt^2 . \sum .m\text{Y}; \quad \sum .md^2z = dt^2 . \sum .m\text{Z};$$

hence

$$\frac{d^2\overline{x}}{dt^2} = \frac{\sum .m\text{X}}{\sum .m}; \quad \frac{d^2\overline{y}}{dt^2} = \frac{\sum .m\text{Y}}{\sum .m}; \quad \frac{d^2\overline{z}}{dt^2} = \frac{\sum .m\text{Z}}{\sum .m}. \qquad \textbf{(22)}$$

These three equations determine the motion of the centre of gravity.

152. Thus the centre of gravity moves as if all the bodies of the system were united in that point, and as if all the forces which act on the system were applied to it.

153. If the mutual attraction of the bodies of the system be the only accelerating force acting on these bodies, the three quantities $\sum m\text{X}$, $\sum m\text{Y}$, $\sum m\text{Z}$, are zero.

Demonstration. This evidently arises from the law of reaction being equal and contrary to action; for if F be the action that an element of the mass m exercises on an element of the mass m', whatever may be the nature of this action, $m'F$ will be the accelerating force with which m is urged by the action of m'; then if f be the mutual distance of m and m', by this action only

$$\text{X} = \frac{m'F\left(x'-x\right)}{f}; \quad \text{Y} = \frac{m'F\left(y'-y\right)}{f}; \quad \text{Z} = \frac{m'F\left(z'-z\right)}{f}. \qquad \textbf{(23)}$$

For the same reasons, the action of m' on m will give

$$\text{X}' = \frac{m\text{F}\left(x-x'\right)}{f}; \quad \text{Y}' = \frac{m\text{F}\left(y-y'\right)}{f}; \quad \text{Z}' = \frac{m\text{F}\left(z-z'\right)}{f};$$

hence

$$m\text{X}+m'\text{X}'=0; \quad m\text{Y}+m'\text{Y}'=0; \quad m\text{Z}+m'\text{Z}'=0;$$

and as all the bodies of the system, taken two and two, give the same results, therefore generally

$$\sum .m\text{X}=0; \quad \sum .m\text{Y}=0; \quad \sum .m\text{Z}=0.$$

154. Consequently

$$\frac{d^2\overline{x}}{dt^2} = 0; \quad \frac{d^2\overline{y}}{dt^2} = 0; \quad \frac{d^2\overline{z}}{dt^2} = 0;$$

and integrating,

$$\overline{x} = at + b; \quad \overline{y} = a't + b'; \quad \overline{z} = a''t + b'';$$

in which a, a', m'; b, b', y', are the arbitrary constant quantities introduced by the double integration. These are equations to straight lines; for, suppose the centre of gravity to begin to move at A, fig. 41, in the direction ox, the distance oA is invariable, and is represented by b; and as at increases with the time t, it represents the straight line AB.

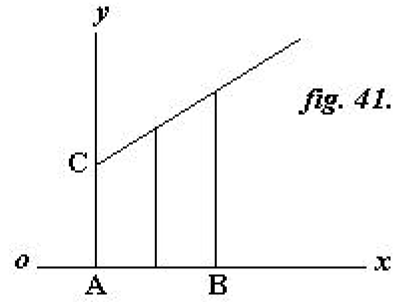


fig. 41.

155. Thus the motion of the centre of gravity in the direction of each axis is a straight line, and by the composition of motions it describes a straight line in space; and as the space it moves over increases with the time, its velocity is uniform; for the velocity, being directly as the element of the space, and inversely as the element of the time, is

$$\sqrt{\left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2 + \left(\frac{d\bar{z}}{dt}\right)^2}\ ;$$

or

$$\sqrt{a^2 + a'^2 + a''^2}\,.$$

Thus the velocity is constant, and therefore the motion uniform.

156. These equations are true, even if some of the bodies, by their mutual action, lose a finite quantity of motion in an instant.

157. Thus, it is possible that the whole solar system may be moving in space; a circumstance which can only be ascertained by a comparison of its position with regard to the fixed stars at very distant periods. In consequence of the proportionality of force to velocity, the bodies of the solar system would maintain their relative motions, whether the system were in motion or at rest.

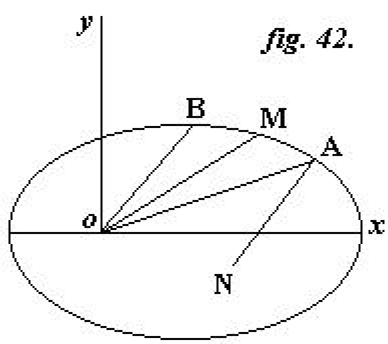


fig. 42.

On the Constancy of Areas

158. If a body propelled by an impulse describe a curve AMB, fig. 42, in consequence of a force of attraction in the point o, that force may be resolved into two, one in the direction of the normal AN, and the other in the direction of the curve or tangent: the first is balanced by the centrifugal force, the second augments or diminishes the velocity of the body; but the velocity is always such that the areas AoM, MoB, described by the radius vector Ao, are proportional to the time; that is, if the body moves from A to M in the same time that it would move from M to B, the area AoM will be equal to the area MoB.

If a system of bodies revolve about any point in consequence of an impulse and a force of attraction directed to that point, the sums of their masses respectively multiplied by the areas described by their radii vectores,[6] when projected on the three co-ordinate planes, are proportional to the time.

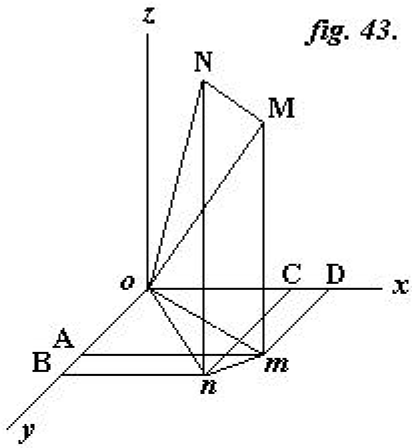


fig. 43.

Demonstration. For if we only consider the areas that are projected on the plane *xoy*, fig. 43, the forces in the direction *oz*, which are perpendicular to that plane, must be zero; hence $Z=0$, $Z'=0$, &c.; and the general equation of the motion of a system of bodies becomes[7]

$$\Sigma . m\left\{X-\frac{d^2x}{dt^2}\right\}dx+\Sigma . m\left\{Y-\frac{d^2y}{dt^2}\right\}dy=0 .$$

If the same assumptions be made here as in article 133, namely,

$$dx = ydv \qquad dy=-xdv$$
$$dx'=y'dv \qquad dy'=-x'dv \text{ , \&c. \&c.,}$$

and if these be substituted in the preceding equation, it becomes, with regard to the plane *xoy*,

$$\Sigma . m\left(\frac{xd^2y-yd^2x}{dt^2}\right)=\Sigma m.\left(xY-yX\right).$$

In the same manner

$$\Sigma m .\left(\frac{zd^2x-xd^2z}{dt^2}\right)=\Sigma m .\left(zX-xZ\right)$$
$$\Sigma m .\left(\frac{yd^2z-zd^2y}{dt^2}\right)=\Sigma m .\left(yZ-zY\right) \qquad \textbf{(24)}$$

are obtained for the motions of the system with regard to the planes *xoz, yoz.* These three equations, together with

$$\Sigma m\cdot\frac{d^2x}{dt^2}=\Sigma mX,\;\; \Sigma m\cdot\frac{d^2y}{dt^2}=\Sigma mY,\;\; \Sigma m\cdot\frac{d^2z}{dt^2}=\Sigma mZ, \qquad \textbf{(25)}$$

are the general equations of the motions of a system of bodies which does not contain a fixed point.

159. When the bodies are independent of foreign forces, and only subject to their reciprocal attraction and to the force at *o*, the sum of the terms

$$m\left\{Xy-Yx\right\}+m'\left\{X'y'-Y'x'\right\},$$

arising from the mutual action of any two bodies in the system, m, m', is zero, by reason of the equality and opposition of action and reaction; and this is true for every such pair as m and m'', m' and m'', &c. If f be the distance of m from o, F the force which urges the body m towards that origin, then

$$\mathrm{X} = -\mathrm{F}\frac{x}{f}, \quad \mathrm{Y} = -\mathrm{F}\frac{y}{f}, \quad \mathrm{Z} = -\mathrm{F}\frac{z}{f}$$

are its component forces; and when substituted in the preceding equations, F vanishes; the same may be shown with regard to m', m'', &c. Hence the equations of areas are reduced to

$$\sum m\left\{\frac{yd^2x - xd^2y}{dt^2}\right\} = 0,$$
$$\sum m\left\{\frac{zd^2x - xd^2z}{dt^2}\right\} = 0,$$
$$\sum m\left\{\frac{yd^2z - zd^2y}{dt^2}\right\} = 0,$$

and their integrals are

$$\begin{aligned} \sum m\{xdy - ydx\} &= cdt \\ \sum m\{zdx - xdz\} &= c'dt \\ \sum m\{ydz - zdy\} &= c''dt \end{aligned} \qquad \textbf{(26)}$$

As the first members or these equations are the sum of the masses of all the bodies of the system, respectively multiplied by the projections of double the areas they describe on the co-ordinate planes, this sum is proportional to the time.

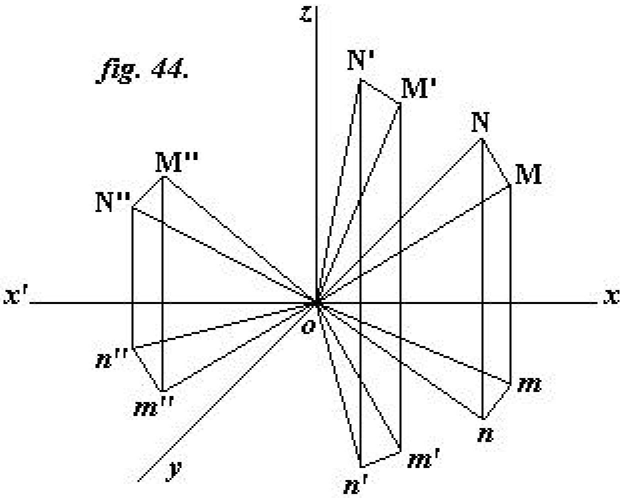
If the centre of gravity be the origin of the co-ordinates, the preceding equations may be expressed thus,

$$cdt = \frac{\sum mm'\{(x'-x)(dy'-dy) - (y'-y)(dx'-dx)\}}{\sum m},$$
$$c'dt = \frac{\sum mm'\{(z'-z)(dx'-dx) - (x'-x)(dz'-dz)\}}{\sum m},$$
$$c''dt = \frac{\sum mm'\{(y'-y)(dz'-dz) - (z'-z)(dy'-dy)\}}{\sum m}.$$

So that the principle of areas is reduced to depend on the co-ordinates of the mutual distances of the bodies of the system.

160. These results way be expressed by a diagram. Let m, m', m'', fig. 44, &c., be a system of bodies revolving about o, the origin of the co-ordinates, in consequence of a central force and a primitive impulse.–Suppose that each of the radii vectores, om, om', om'', &c., describes the indefinitely small areas, MoN, M′oN′, &c., in an indefinitely small time, represented by dt; and let mon, $m'on'$, &c., be the projections of these areas on the plane xoy. Then the equation

fig. 44.



$$\sum m\{xdy - ydx\} = cdt\,,$$

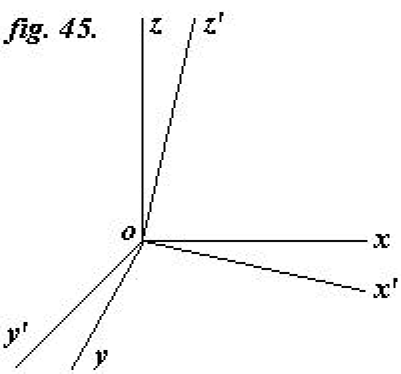
shows that the sum of the products of twice the area mon, by the mass m, twice the area $m'on'$ by the mass m', twice $m''on''$ by the mass m'', &c., is proportional to the element of the time in which they are described: whence it follows that the sum of the projections of the areas, each multiplied by the corresponding mass, is proportional to the finite time in which they are described. The other two equations express similar results for the areas projected on the planes xoz, yoz.

161. The constancy of areas is evidently true for any plane whatever, since the position of the co-ordinate planes is arbitrary. The three equations of areas give the space described by the bodies on each co-ordinate plane in value of the time: hence, if the time be known or assumed, the corresponding places of the bodies will be found on the three planes, and from thence their true positions in space may be determined, since that of the co-ordinate planes is supposed to be known. It was shown, in article 132, that

$$\sum m\{xdy - ydx\},$$
$$\sum m\{zdx - xdz\},$$
$$\sum m\{zdy - ydz\},$$

are the pressures of the system, tending to make it turn round each of the axes of the co-ordinates: hence the principle of areas consists in this–that the sum of the rotatory pressures which cause a system of bodies to revolve about a given point, is zero when the system is in equilibrio, and proportional to the time when the system is in motion.

fig. 45.



162. Let us endeavour to ascertain whether any planes exist on which the sums of the areas are zero when the system is in motion. To solve this problem it is necessary to determine one set of coordinates in values of another.

163. If ox, oy, oz, fig. 45, be the co-ordinates of a point m, it is required to determine the position of m by means of

ox', oy', oz', three new rectangular co-ordinates, having the same origin as the former.

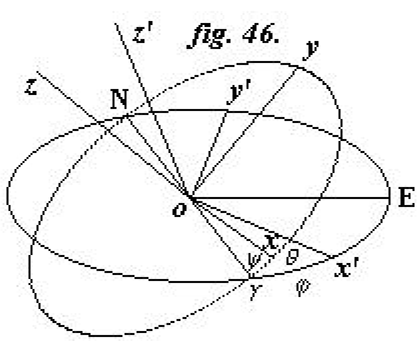
We shall find a value of ox or x first. Now,

$$ox : ox' :: 1 : \cos xox' \quad \text{or} \quad x' = x\cos xox'.$$
$$ox : oy' :: 1 : \cos xoy' \quad \text{or} \quad y' = x\cos xoy'.$$
$$ox : oz' :: 1 : \cos xoz' \quad \text{or} \quad z' = x\cos xoz'.$$

If the sum of these quantities be taken, after multiplying the first by $\cos xox'$, the second by $\cos xoy'$, and the third by $\cos xoz'$, we shall have

$$x'\cos xox' + y'\cos xoy' + z'\cos xoz' = x\{\cos^2 xox' + \cos^2 xoy' + \cos^2 xoz'\} = x.$$

Let $o\gamma$, fig. 46, be the intersection of the old plane xoy with the new $x'oy'$; and let θ be the inclination of these two planes; also let γox, $\gamma ox'$, be represented by ψ and φ. Values of the cosines of xox', xoy', xoz', must be found in terms of θ, ψ, and φ. In the right-angled triangle $\gamma xx'$, the sides γx, $\gamma x'$, are ψ and φ, and the angle opposite the side xx' is θ:–hence, by spherical trigonometry,



$$\cos xox' = \cos\theta \sin\varphi \sin\psi + \cos\psi \cos\varphi.$$

This equation exists, whatever the values of φ and ψ may be; hence, if $\varphi + 90^\circ$ be put for φ, the line ox' will take the place oy', the angle xox' will become xoy', and the preceding equation will give

$$\cos xoy' = \cos\theta \sin\psi \cos\varphi - \cos\psi \sin\varphi.$$

Cos xoz' is found from the triangle whose three sides are the arcs intercepted by the angles $\gamma oz'$, γox, and xoz'. The angle opposite to the last side is

$$90^\circ - \theta, \ \gamma oz' = 90^\circ, \ \gamma ox = \psi,$$

then the general equation becomes

$$\cos xoz' = \sin\theta \sin\psi.$$

If these expressions for the cosines be substituted in the value of x, it becomes

$$x = x'\{\cos\theta \sin\varphi \sin\psi + \cos\psi \cos\varphi\} + y'\{\cos\theta \cos\varphi \sin\psi - \cos\psi \sin\varphi\} + z'\sin\theta \sin\psi.$$

In the same manner, the values of y and z are found to be

$$y = x'\{\cos q \sin y \sin f - \sin y \cos f\} + y'\{\cos q \cos y \cos f + \sin y \sin f\} + z'\{\sin q \cos y\}$$
$$z = -x'\{\sin q \sin f\} - y'\{\sin q \cos f\} + z'\cos q \ .$$

By substituting these values of $x, y, z,$ in any equation, it will be transformed from the planes xoy, xoz, yoz, to the new planes $x'o y'$, $x'o z'$, $y'o z'$.

164. We have now the means of ascertaining whether, among the infinite number of co-ordinate planes whose origin is in o, the centre of gravity of a system of bodies, there be any on which the sums of the areas are zero. This may be known by substituting the preceding values of $x, y, z,$ and their differentials in the equations of areas: for the angles q, y, and f being arbitrary, such values may be assumed for two of them as will make the sums of the projected areas on two of the co-ordinate planes zero; and if there be any incongruity in this assumption, it will appear in the determination of the third angle, which in that case would involve some absurdity in the areas on the third plane. That, however, is by no means the case, for the sum of the areas on the third plane is then found to be a maximum. If the substitution be made, and the angles y and q so assumed that

$$\sin q \sin y = \frac{c''}{\sqrt{c^2 + c'^2 + c''^2}}, \quad \sin q \cos y = \frac{-c'}{\sqrt{c^2 + c'^2 + c''^2}},$$

it follows that

$$\cos q = \frac{c}{\sqrt{c^2 + c'^2 + c''^2}},$$

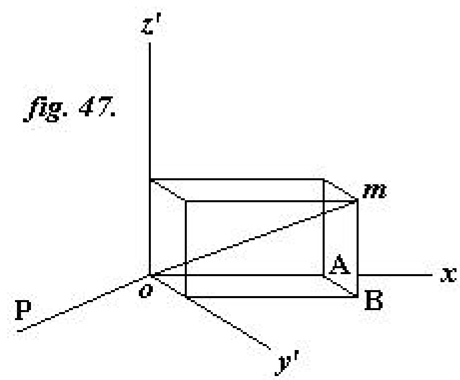
whence

$$\sum m \frac{x'dy' - y'dx'}{dt} = \sqrt{c^2 + c'^2 + c''^2}, \quad \sum m \frac{x'dz' - z'dx'}{dt} = 0, \quad \sum m \frac{y'dz' - z'dy'}{dt} = 0. \qquad \textbf{(27)}$$

Thus, in every system of revolving bodies, there does exist a plane, on which the sum of the projected areas is a maximum; and on every plane at right angles to it, they are zero. One plane alone possesses that property.

165. If the attractive force at o were to cease, the bodies would move by the primitive impulse alone, and the principle of areas would be also true in this case; it even exists independently of any abrupt changes of motion or velocity, among the bodies; and also when the centre of gravity has a rectilinear motion in space. Indeed it follows as a matter of course, that all the properties which have been proved to exist in the motions of a system of bodies, whose centre of gravity is at rest, must equally exist, if that point has a uniform and rectilinear motion in space, since experience shows that the relative motions of a system of bodies, are[8] independent of any motion common to them all.

fig. 47.



Demonstration. However, that will readily appear, if $\bar{x}$, $\bar{y}$, $\bar{z}$, be assumed, as the co-ordinates of o, the moveable centre of gravity estimated from a fixed point P, fig. 47, and if oA; AB, Bm, or x', y', z', be the co-ordinates of m, one of the bodies of the system with regard to the moveable point o. Then the co-ordinates of m relatively to P will be $\bar{x}+x'$, $\bar{y}+y'$, $\bar{z}+z'$. If these be put instead of x, y, z, in the different equations relative to the motions of a system, by attending to the properties of the centre of gravity, $\bar{x}$, $\bar{y}$, $\bar{z}$, vanish from these equations, which then become independent of them. If $\bar{x}+x'$, $\bar{y}+y'$, $\bar{z}+z'$ be put for x, y, z, in equations (25), they become[9]

$$\sum .m\left\{d^2\bar{x}+d^2x'\right\}-\sum m\mathrm{X}dt^2=0.$$
$$\sum .m\left\{d^2\bar{y}+d^2y'\right\}-\sum m\mathrm{Y}dt^2=0.$$
$$\sum .m\left\{d^2\bar{z}+d^2z'\right\}-\sum m\mathrm{Z}dt^2=0.$$

But when the centre of gravity has a rectilinear and uniform motion in space, it has been shown, that

$$\frac{d^2\bar{x}}{dt^2}=0;\quad \frac{d^2\bar{y}}{dt^2}=0;\quad \frac{d^2\bar{z}}{dt^2}=0;$$

which reduces the preceding equations to their original form, namely,[10]

$$\sum m\cdot\frac{d^2x'}{dt^2}=\sum m\mathrm{X},\quad \sum m\cdot\frac{d^2y'}{dt^2}=\sum m\mathrm{Y},\quad \sum m\cdot\frac{d^2z'}{dt^2}=\sum m\mathrm{Z}.$$

If the same substitution be made in[11]

$$\sum m\left(\frac{xd^2y-yd^2x}{dt^2}\right)=\sum m\left(x\mathrm{Y}-y\mathrm{X}\right)$$

it becomes

$$\frac{\bar{x}\sum .md^2y'-\bar{y}\sum md^2x'}{dt^2}+\sum m\cdot\frac{x'd^2y'-y'd^2x'}{dt^2}=\sum m\,.\left(\mathrm{Y}x'-\mathrm{X}y'\right)+\bar{x}\,.\sum m\mathrm{Y}-\bar{y}\,.\sum m\mathrm{X}\,.$$

But in consequence of the preceding equations it is reduced to[12]

$$\sum .m\cdot\left\{\frac{x'd^2y'-y'd^2x'}{dt^2}\right\}=\sum .m\cdot\left(x'\mathrm{Y}-y'\mathrm{X}\right).$$

In the same manner it may be shown that

$$\sum .m\cdot\left\{\frac{z'd^2x'-x'd^2z'}{dt^2}\right\}=\sum .m\,.\left(z'\mathrm{X}-x'\mathrm{Z}\right),$$

$$\Sigma.m.\left\{\frac{y'd^2z'-z'd^2y'}{dt^2}\right\}=\Sigma.m.\left(y'\mathrm{Z}-z'\mathrm{Y}\right).$$

Thus the equations that determine the motions of a system of bodies are the same, whether the centre of gravity be at rest, or moving uniformly in a straight line; consequently the principles of Impetus, of Least Action, and of the Conservation of Areas, exist in either case.

166. Let the effect produced by the motion of the centre of gravity on the position of the plane for which the areas are a maximum, be now determined.

If $\bar{x}+x$, $\bar{y}+y$, $\bar{z}+z$, be put for x, y, z, in equations (26), they will retain the same form, namely,

$$\Sigma m\{x'dy'-y'dx'\}=cdt,$$
$$\Sigma m\{z'dx'-x'dz'\}=c'dt,$$
$$\Sigma m\{y'dz'-z'dy'\}=c''dt;$$

for, in consequence of the rectilinear motion of the origin,

$$\bar{x}d\bar{y}-\bar{y}d\bar{x}=0,\quad \bar{z}d\bar{x}-\bar{x}d\bar{z}=0,\quad \bar{y}d\bar{z}-\bar{z}d\bar{y}=0.$$

And as the position of the plane in question is determined by the constant quantities c, c', and c'', it will always remain parallel to itself during the motion of the system; on that account it is called the Invariable Plane.

167. Thus, when there are no foreign forces acting on the system, the centre of gravity either remains at rest, or moves uniformly in a straight line; and if that point be assumed as the origin of the co-ordinates; the principles of the conservation of areas and living forces will exist with regard to it; and the invariable plane, always passing through that point, will remain parallel to itself, and will be carried along with the centre of gravity in the general motion of the system.

On the Motion of a System of Bodies in all possible Mathematical relations between Force and Velocity

168. In nature, force is proportional to velocity; but as a matter of speculation, Laplace[13] has investigated the motions of a system of bodies in every possible relation between these two quantities. It is rather singular that such an hypothesis should involve no contradiction; on the contrary, principles similar to the preservation of impetus, the constancy of areas, the motion of the centre of gravity, and the least action, actually exist.

Notes

[1] V^2 *reads* V_2 in the 1st edition.

[2] This *reads* $V_2 = C+2v$ in the 1st edition.

[3] $d\overline{x}$ *reads* dx in the 1st edition (published erratum).

[4] *shewn*. Archaic for *shown*.

[5] This *reads* $\sum \cdot md^2x = dt^2 \cdot \sum \cdot m\mathrm{X};\ \ \sum \cdot md^2y = dt^2 \cdot \sum m\mathrm{Y};\ \ \sum \cdot md^2z = dt^2 \sum \cdot m\mathrm{Z}$ in 1st edition.

[6] *vectores*. Archaic plural for *vector*.

[7] First term below *reads* $\sum m\left\{\mathrm{X} - \frac{d^2x}{dt^2}\right\}dx$ in the1st edition.

[8] This *reads* "is" in the 1st edition.

[9] Second member of third equation below *reads* $-\sum \cdot m\mathrm{Z}dt^2$ in the 1st edition.

[10] The first term in first equation below *reads* $\sum \cdot m\frac{d^2x'}{dt^2} = \sum m\mathrm{X}$ in the 1st edition.

[11] This is the first of equations (24) which we reproduce as formulated earlier.

[12] The closing bracket on the right hand side of the following equation is omitted in the 1st edition.

[13] See note 4, *Introduction*.