

BOOK II

CHAPTER II

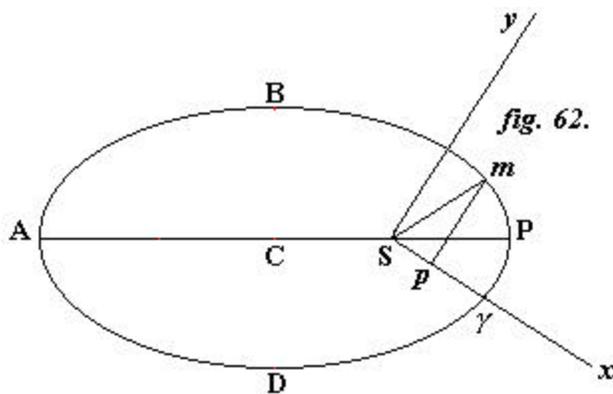
ON THE LAW OF UNIVERSAL GRAVITATION, DEDUCED FROM OBSERVATION

309. THE three laws of Kepler furnish the data from which the principle of gravitation is established, namely:—

- i. That the radii vectores of the planets and comets describe areas proportional to the time.
- ii. That the orbits of the planets and comets are conic sections, having the sun in one of their foci.
- iii. That the squares or the periodic times of the planets are proportional to the cubes of their mean distances from the sun.

310. It has been shown, that if the law of the force which acts on a moving body be known, the curve in which it moves may be found; or, if the curve in which the body moves be given, the law of the force may be ascertained. In the general equation of the motion of a body in article 144, both the force and the path of the body are indeterminate; therefore in applying that equation to the motion of the planets and comets, it is necessary to know the orbits in which they move, in order to ascertain the nature of the force that acts on them.

311. In the general equation of the motion of a body, the forces acting on it are resolved into three component forces, in the direction of three rectangular axes; but as the paths of the planets, satellites, and comets, are proved by the observations of Kepler to be conic sections, they always move in the same plane: therefore the component force in the direction perpendicular to that plane is zero, and the other two component forces are in the plane of the orbit.



312. Let AmP , fig. 62, be the elliptical orbit of a planet m , having the centre of the sun in the focus S , which is also assumed as the origin of the co-ordinates. The imaginary line Sm joining the centre of the sun and the centre of the planet is the radius vector. Suppose the two component forces to be in the direction of the axes Sx , Sy , Sz , then the component force Z , is zero; and as the body is free to move in every direction, the virtual velocities dx , dy are zero, which divides the general equation of motion in article 144 into

$$\frac{d^2x}{dt^2} = X; \quad \frac{d^2y}{dt^2} = Y;$$

giving a relation between each component force, the space that it causes the body to describe on ox , or oy , and the time. If the first of these two equations be multiplied by $-y$, and added to the second multiplied by x , their sum will be

$$\frac{d(xdy - ydx)}{dt^2} = Yx - Xy.$$

But $x dy - y dx$ is double the area that the radius vector of the planet describes round the sun in the instant dt . According to the first law of Kepler, this area is proportional to the time, so that

$$x dy - y dx = c dt;$$

and as c is a constant quantity,

$$\frac{d(xdy - ydx)}{dt^2} = 0,$$

therefore

$$Yx - Xy = 0,$$

whence

$$X : Y :: x : y;$$

so that the forces X and Y are in the ratio of x to y , that is as Sp to pm , and thus their resulting force mS passes through S , the centre of the sun. Besides, the curve described by the planet is concave towards the sun, whence the force that causes the planet to describe that curve, tends towards the sun. And thus the law of the areas being proportional to the time, leads to this important result,—that the force which retains the planets and comets in their orbits, is directed towards the centre of the sun.

313. The next step is to ascertain the law by which the force varies at different distances from the sun, which is accomplished by the consideration, that these bodies alternately approach and recede from him at each revolution; the nature of elliptical motion, then, ought to give that law. If the equation

$$\frac{d^2x}{dt^2} = X$$

be multiplied by dx , and

$$\frac{d^2y}{dt^2} = Y,$$

by dy , their sum is

$$\frac{dx d^2 x + dy d^2 y}{dt^2} = X dx + Y dy ,$$

and its integral is

$$\frac{d^2 x + d^2 y}{dt^2} = 2 \int (X dx + Y dy) ,$$

the constant quantity being indicated by the integral sign. Now the law of areas gives

$$dt = \frac{xdy - ydx}{c} ,$$

which changes the preceding equation to

$$\frac{c^2 (dx^2 + dy^2)}{(xdy - ydx)^2} = 2 \int (X dx + Y dy) . \quad (82)$$

In order to transform this into a polar equation, let r represent the radius vector Sm , fig. 62, and v the angle mSg , then

$$Sp = x = r \cos v; pm = y = r \sin v; \text{ and } r = \sqrt{x^2 + y^2}$$

whence

$$dx^2 + dy^2 = r^2 dv^2 + dr^2, \quad xdy - ydx = r^2 dv ;$$

and if the resulting force of X and Y be represented by F , then

$$F : X :: Sm : Sp :: 1 : \cos v ;$$

hence

$$X = -F \cos v ;$$

the sign is negative, because the force F in the direction mS , tends to diminish the co-ordinates; in the same manner it is easy to see that

$$Y = -F \sin v; F = \sqrt{X^2 + Y^2}; \text{ and } X dx + Y dy = -F dr ;$$

so that the equation (82) becomes

$$0 = \frac{c^2 \{ r^2 dv^2 + dr^2 \}}{r^4 dv^2} + 2 \int F dr . \quad (83)$$

Whence¹

$$dv = \frac{cdr}{r\sqrt{-c^2 - 2r^2} \int Fdr}.$$

314. If the force F be known in terms of the distance r , this equation will give the nature of the curve described by the body. But the differential of equation (83) gives

$$F = \frac{c^2}{r^3} - \frac{c^2}{2} d \left\{ \frac{dr^2}{r^4 dv^2} \right\}. \quad (84)$$

Thus a value of the resulting force F is obtained in terms of the variable radius vector Sm , and of the corresponding variable angle mSg ; but in order to have a value of the force F in terms of mS alone, it is necessary to know the angle gSm in terms of Sm . The planets move in ellipses, having the sun in one of their foci; therefore let ν represent the angle gSP , which the greater axis AP makes with the axes of the co-ordinates Sx , and let v be the angle gSm . Then if $\frac{CS}{CP}$, the ratio of the eccentricity to the greater axis be e , and half² the greater axis $CP=a$, the polar equation of conic sections is

$$r = \frac{a(1-e^2)}{1+e \cos(\nu - \nu)},$$

which becomes a parabola when $e=1$, and a infinite; and a hyperbola when e is greater than unity and a negative. This equation gives a value of r in terms of the angle gSm or v , and thence it may be found that

$$\frac{dr^2}{r^4 dv^2} = \frac{2}{ar(1-e^2)} - \frac{1}{r^2} - \frac{1}{a^2(1-e^2)}$$

which substituted in equation (84) gives

$$F = \frac{c^2}{a(1-e^2)} \cdot \frac{1}{r^2}.$$

The coefficient $\frac{c^2}{a(1-e^2)}$ is constant, therefore F varies inversely as the square of r or

Sm . Wherefore the orbits of the planets and comets being conic sections, the force varies inversely as the square of the distance of these bodies from the sun.

Now as the force F varies inversely as the square of the distance, it may be represented by $\frac{h}{r^2}$, in which h is a constant coefficient, expressing the intensity of the force. The equation of conic sections will satisfy equation (84) when $\frac{h}{r^2}$ is put for F ; whence as

$$h = \frac{c^2}{a(1 - e^2)}$$

forms an equation of condition between the constant quantities a and e , the three arbitrary quantities a , e , and \mathbf{v} , are reduced to two; and as equation (83) is only of the second order, the finite equation of conic sections is its integral.

315. Thus, if the orbit be a conic section, the force is inversely as the square of the distance; and if the force varies inversely as the square of the distance, the orbit is a conic section. The planets and comets therefore describe conic sections in virtue of a primitive impulse and an accelerating force directed to the centre of the sun, and varying according to the preceding law, the least deviation from which would cause them to move in curves of a totally different nature.

316. In every orbit the point P , fig. 63, which is nearest the sun, is the perihelion, and in the ellipse the point A farthest from the sun is the aphelion. SP is the perihelion distance of the body from the sun.

317. A body moves in a conic section with a different velocity in every point of its orbit, and with a perpetual tendency to fly off in the direction of the tangent, but this tendency is counteracted by the attraction of the sun. At the perihelion, the velocity of a planet is greatest; therefore its tendency to leave the sun exceeds the force of attraction: but the continued action of the sun diminishes the velocity as the distance increases; at the aphelion the velocity of the planet is least: therefore its tendency to leave the sun is less than the force of attraction which increases the velocity as the distance diminishes, and brings the planet back towards the sun, accelerating its velocity so much as to overcome the force of attraction, and carry the planet again to the perihelion. This alternation is continually repeated.

318. When a planet is in the point B , or D , it is said to be in quadrature, or at its mean distance from the sun. In the ellipse, the mean distance, SB or SD , is equal to CP , half the greater axis; the eccentricity is CS .

319. The periodic time of a planet is the time in which it revolves round the sun, or the time of moving through 360° . The periodic time of a satellite is the time in which it revolves about its primary.

320. From the equation³

$$F = \frac{c^2}{a(1-e^2)} \cdot \frac{1}{r^2},$$

it may be shown, that the force F varies, with regard to different planets, inversely as the square of their respective distances from the sun. The quantity $2a(1-e^2)$ is $2SV$, the parameter of the orbit, which is invariable in any one curve, but is different in each conic section. The intensity of the force depends on

$$\frac{c^2}{a(1-e^2)} \text{ or } \frac{c^2}{SV},$$

which may be found by Kepler's laws. Let T represent the time of the revolution of a planet; the area described by its radius vector in this time is the whole area of the ellipse, or

$$pa^2 \cdot \sqrt{1-e^2}.$$

where $p = 3.14159$ the ratio of the circumference to the diameter. But the area described by the planet during the indefinitely small time dt , is $\frac{1}{2}cdt$; hence the law of Kepler gives

$$\frac{1}{2}cdt : pa^2\sqrt{1-e^2} :: dt : T ;$$

whence

$$c = \frac{2pa^2\sqrt{1-e^2}}{T}. \quad (85)$$

But, by Kepler's third law, the squares of the periodic times of the planets are proportional to the cubes of their mean distances from the sun; therefore

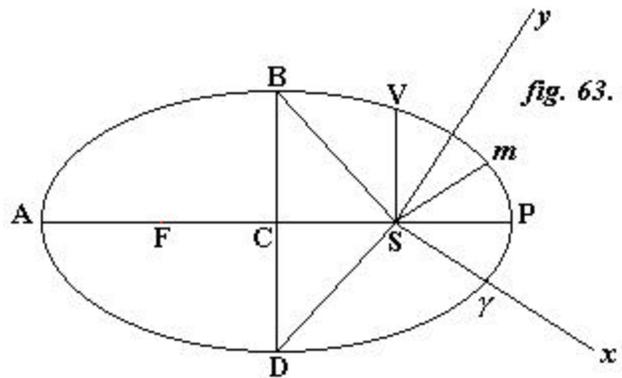
$$T^2 = k^2 a^3,$$

k being the same for all the planets. Hence

$$c = \frac{2p\sqrt{a(1-e^2)}}{k};$$

but $2a(1-e^2)$ is $2SV$, the parameter of the orbit.

Therefore, in different orbits compared together, the values of c are as the areas traced by the radii vectores in equal times; consequently those areas are proportional to the square roots of the parameters of the orbits, either of planets or comets. If this value of c be put in⁴



$$F = \frac{c^2}{a(1-e^2)} \cdot \frac{1}{r^2}$$

it becomes⁵

$$F = \frac{4p^2}{k^2} \cdot \frac{1}{r^2} = h \cdot \frac{1}{r^2};$$

in which $\frac{4p^2}{k^2}$ or h , is the same for all planets or comets; the force, therefore, varies inversely as the square of the distance of each from the centre of the sun: consequently, if all these bodies were placed at equal distances from the sun, and put in motion at the same instant from a state of rest, they would move through equal spaces in equal times; so that all would arrive at the sun at the same instant,—properties first demonstrated geometrically by Newton from the laws of Kepler.

321. That the areas described by comets are proportional to the square roots of the parameters of their orbits, is a result of theory more sensibly verified by observation than any other of its consequences. Comets are only visible for a short time, at most a few months, when they are near their perihelia; but it is difficult to determine in what curve they move, because a very eccentric ellipse, a parabola, and hyperbola of the same perihelion distance coincide through a small space on each side of the perihelion. The periodic time of a comet cannot be known from one appearance. Of more than a hundred comets, whose orbits have been computed, the return of only three has been ascertained. A few have been calculated in very elliptical orbits; but in general it has been found, that the places of comets computed in parabolic orbits agree with observation: on that account it is usual to assume, that comets move in parabolic curves.

322. In a parabola the parameter is equal to twice the perihelion distance, or

$$a(1-e^2) = 2D;$$

hence, for comets,

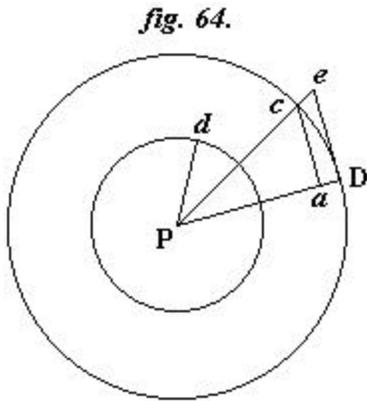
$$c = \frac{2p}{k} \sqrt{2D}.$$

For, in this case, $e = 1$ and a , is infinite; therefore, in different parabolae, the areas described in different times are proportional to the square roots of their perihelion distances. This affords the means of ascertaining how near a comet approaches to the sun. Five or six comets seem to have hyperbolic orbits; consequently they could only be once visible, in their transit through the system to which we belong, wandering in the immensity of space, perhaps to visit other suns and other systems.

It is probable that such bodies do exist in the infinite variety of creation, though their appearance is rare. Most of the comets that we have seen, however, are thought to move in extremely eccentric ellipses, returning to our system after very long intervals. Two hundred years have not elapsed since comets were observed with accuracy, a time which is probably greatly exceeded by the enormous periods of the revolutions of some of these bodies.

323. The discovery of the⁶ laws of Kepler, deduced from the observations of Tycho Brahe,⁷ and from his own observations of Mars, form an era of vast importance in the science of astronomy, being the bases on which Newton founded the universal principle of gravitation: they lead us to regard the centre of the sun as the focus of an attractive force, extending to an infinite distance in all directions, decreasing as the squares of the distance increase. Each law discloses a particular property of this force. The areas described by the radius vector of each planet or comet, being proportional to the time employed in describing them, shows that the principal force which urges these bodies, is always directed towards the centre of the sun. The ellipticity of the planetary orbits, and the nearly parabolic motion of the comets, prove that for each planet and comet this force is reciprocally as the square of the distance from the sun; and, lastly, the squares of the periodic times, being proportional to the cubes of the mean distances, proves that the areas described in equal times by the radius vector of each body in the different orbits, are proportional to the square roots of the parameters—a law which is equally applicable to planets and comets.

324. The satellites observe the laws of Kepler in moving round their primaries, and gravitate towards the planets inversely as the square of their distances from their centre; but they must also gravitate towards the sun, in order that their relative motions round their planets may



be the same as if the planets were at rest. Hence the satellites must gravitate towards their planets and towards the sun inversely as the squares of the distances. The eccentricity of the orbits of the two first satellites of Jupiter is quite insensible; that of the third inconsiderable; that of the fourth is evident. The great distance of Saturn has hitherto prevented the eccentricity of the orbits of any of its satellites from being perceived, with the exception of the sixth. But the law of the gravitation of the satellites of Jupiter and Saturn is derived most clearly from this ratio,—that, for each system of satellites, the squares of their periodic times are as the cubes of their mean distances from the centres of their respective planets. For, imagine a satellite to describe a circular orbit, with a radius

$PD = a$, fig. 64, its mean distance from the centre of the planet. Let T be the duration of a sidereal revolution of the satellite, then $3.14159 = p$, being the ratio of the circumference to the diameter, $a \cdot \frac{2p}{T}$ will be the very small arc Dc that the satellite describes in a second. If the attractive force of the planet were to cease for an instant, the satellite would fly off in the tangent De , and would be farther from the centre of the planet by a quantity equal to aD , the versed sine of the arc Dc . But the value of the versed sine is

$$a \cdot \frac{2p^2}{T^2},$$

which is the distance that the attractive force of the planet causes the satellite to fall through in a second.

Now, if another satellite be considered, whose mean distance is $PD = a'$, and T' , the duration of its sidereal revolution, its deflection will be⁸ $a' \cdot \frac{2p^2}{T'^2}$ in a second; but if F and F' be the attractive forces of the planet at the distances PD and Pd , they will evidently be proportional to the quantities they make the two satellites fall through in a second; hence

$$F : F' :: a \frac{2p^2}{T^2} : a' \frac{2p^2}{T'^2},$$

or

$$F : F' :: \frac{a}{T^2} : \frac{a'}{T'^2};$$

but the squares of the periodic times are as the cubes of the mean distances; hence

$$T^2 : T'^2 :: a^3 : a'^3;$$

Thus the satellites gravitate to their primaries inversely as the square of the distance.

325. As the earth has but one satellite, this comparison cannot be made, and therefore the ellipticity of the lunar orbit is the only celestial phenomenon by which we can know the law of the moon's attractive force. If the earth and the moon were the only bodies in the system, the moon would describe a perfect ellipse about the earth; but, in consequence of the action of the sun, the path of the moon is sensibly disturbed, and therefore is not a perfect ellipse; on this account some doubts may arise as to the diminution of the attractive force of the earth as the inverse square of the distance.

The analogy, indeed, which exists between this force and the attractive force of the sun, Jupiter, and Saturn, would lead to the belief that it follows the same law, because the solar attraction acts equally on all bodies placed at the same distance from the sun, in the same manner that terrestrial gravitation causes all bodies in vacuo to fall from equal heights in equal times. A projectile thrown horizontally from a height, falls to the earth after having described a parabola. If the force of projection were greater, it would fall at a greater distance; and if it amounted to 30,772.4 feet in a second, and were not resisted by the air, it would revolve like a satellite about the earth, because its centrifugal force would then be equal to its gravitation. This body would move in all respects like the moon, if it were projected with the same force, at the same height.

It may be proved, that the force which causes the descent of heavy bodies at the surface of the earth, diminished in the inverse ratio of the square of the distance, is sufficient to retain the moon in her orbit, but this requires a knowledge of the lunar parallax.

On Parallax

326. Let m , fig. 65, be a body in its orbit, and C the centre of the earth, assumed to be spherical. A person on the surface of the earth, at E , would see the body m in the direction EmB ; but the body would appear, in the direction CmA , to a person in C , the centre of the earth. The

angle CmE , which measures the difference of these directions, is the parallax of m . If z be the zenith of an observer at E , the angle zEm , called the zenith distance of the body, may be measured; hence mEC is known, and the difference between zEm and zCm is equal to CmE , the parallax, then if $CE = R$, $Cm = r$, and $zEm = z$, [then]⁹

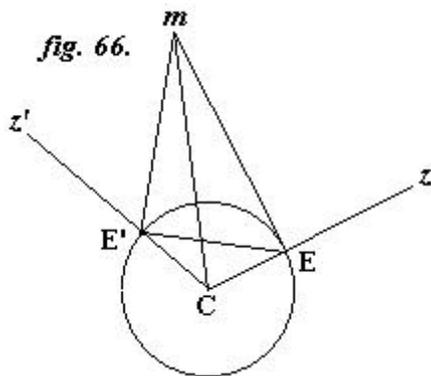
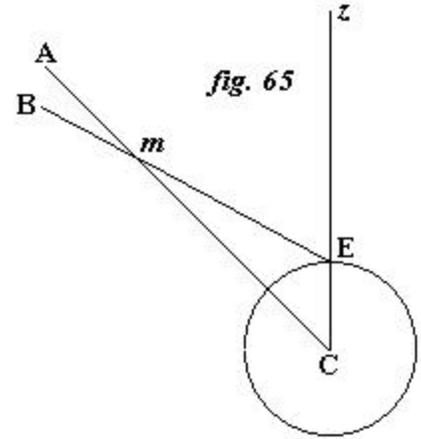
$$\sin CmE = \frac{R}{r} \sin z;$$

hence, if CE and Cm remain the same, the sine of the parallax, CmE , will vary as the sine of the zenith distance zEm ; and when $zEm = 90^\circ$, as in fig. 67,

$$\sin P = \frac{R}{r};$$

P being the value of the angle CmE in this case; then the parallax is a maximum, for Em is tangent to the earth, and, as the body m is seen in the horizon, it is called the horizontal parallax; hence the sine of the horizontal parallax is equal to the terrestrial radius divided by the distance of the body from the centre of the earth.

327. The length of the mean terrestrial radius is known, the horizontal parallax may be determined by observation, therefore the distance of m from the centre of the earth is known. By this method the dimensions of the solar system have been ascertained with great accuracy. If the distance be very great compared with the diameter of the earth, the parallax will be insensible. if CmE were an angle of the fourth of a second, it would be inappreciable; an arc of $1'' = 0.000004848$ of the radius, the fourth of a second is therefore $0.000001212 = \frac{1}{825,082}$; and thus, if a body be distant from the earth by 825,082 of its semidiameters, or 3,265,660,000 miles, it will be seen in the same position from every point of the earth's surface. The parallax of all the celestial bodies is very small: even that of the moon at its maximum does not much exceed 1° .



328. P being the horizontal parallax, let p be the parallax EmC , fig. 66, at any height. When P is known, p may be found, and the contrary, for if $\frac{R}{r}$ be eliminated, then $\sin p = \sin P \sin z$, and when P is constant, $\sin p$ varies as $\sin z$.

329. The horizontal parallax is determined as follows: let E and E' , fig. 66, be two places on the same meridian of the earth's surface; that is, which contemporaneously have the same noon. Suppose the latitudes of these two places to be

perfectly known; when a body m is on the meridian, let its zenith distances $zEm = z$, $z'E'm = z'$, be measured by two observers in E and E' . Then ECE' , the sum of the latitudes, is known, and also the angles CEm , $CE'm$; hence EmE' , EmC , and $E'mC$ may be determined; for P is so small, that it may be put for its sine; therefore

$$\sin p = P \sin z, \quad \sin p' = P \sin z';$$

and as p and p' are also very small,¹⁰

$$p + p' = P \{ \sin z + \sin z' \}.$$

Now, $p + p'$ is equal to the angle EmE' , under which the chord of the terrestrial arc EE' , which joins the two observers, would be seen from the centre of m , and it is the fourth angle of the quadrilateral $CEmE'$. But

$$CEm = 180^\circ - z, \quad CE'm = 180^\circ - z',$$

and if

$$ECm + E' Cm = f,$$

then

$$180^\circ - z + 180^\circ - z' + p + p' + f = 360^\circ;$$

hence

$$p + p' = z + z' - f;$$

therefore the two values of $p + p'$ give

$$P = \frac{z + z' - f}{\sin z + \sin z'},$$

which is the horizontal parallax of the body, when the observers are on different sides of Cm ; but when they are on the same side,

$$P = \frac{z - z' - f}{\sin z - \sin z'}.$$

It requires a small correction, since the earth, being a spheroid, the lines¹¹ zE , $z'E'$ do not pass through C , the centre of the earth.

The parallax of the moon and of Mars were determined in this manner, from observations made by Lacaille^{12 13} at the Cape of Good Hope, in the southern hemisphere; and by Wargesten at Stockholm, which is nearly on the same meridian in the northern hemisphere.

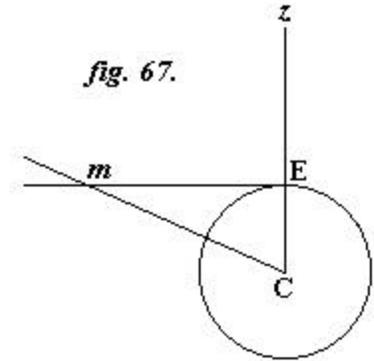
330. The horizontal parallax varies with the distance of the body from the earth; for it is evident that the greater the distance, the less the parallax. It varies also with the parallels of terrestrial latitude, the earth, being a spheroid, the length of the radius decreases from the equator

to the poles. It is on this account that, at the mean distance of the moon, the horizontal parallax observed in different latitudes varies; proving the elliptical figure of the earth. The difference between the mean horizontal parallax at the equator and at the poles, from this cause, is $10''.3$.

331. In order to obtain a value of the moon's horizontal parallax, independent of these inequalities, the horizontal parallax is chosen at the mean distance of the moon from the earth, and on that parallel of terrestrial latitude, the square of whose sine is $\frac{1}{3}$, because the attraction of the earth upon the corresponding points of its surface is nearly equal to the mass of the earth, divided by the square of the mean distance of the moon from the earth. This is called the constant part of the horizontal parallax. The force which retains the moon in her orbit may now be determined.

Force of Gravitation at the Moon

332. If the force of gravity be assumed to decrease as the inverse square of the distance, it is clear that the force of gravity at E, fig. 67, would be, to the same force at *m*, the distance of the moon, as the square of *Cm* to the square of *CE*; but *CE* divided by *Cm* is the sine of the horizontal parallax of the moon, the constant part of which is found by observation to be $57'4.17''$ in the latitude in question; hence the force of gravity, reduced to the distance of the moon, is equal to the force of gravity at E on the earth's surface, multiplied by $\sin^2 57'4''.17$, the square of the sine of the constant part of the horizontal parallax.



Since the earth is a spheroid, whose equatorial diameter is greater than its polar diameter, the force of gravity increases from the equator to the poles; but it has the same intensity in all points of the earth's surface in the same latitude.

Now the space through which a heavy body would fall during a second in the latitude the square of whose sine is $\frac{1}{3}$, has been ascertained by experiments with the pendulum to be 16.0697 feet; but the effect of the centrifugal force makes this quantity less than it would otherwise be, since that force has a tendency to make bodies fly off from the earth. At the equator it is equal to the 288th part of gravity; but as it decreases from the equator to the poles as the square of the sine of the latitude, the force of gravity in that latitude the square whose sine is $\frac{1}{3}$, is only diminished by two-thirds of $\frac{1}{288}$ or by its 432nd part. But the 432nd part of 16.0697 is 0.0372, and adding it to 16.0697, the whole effect of terrestrial gravity in the latitude in question is 16.1069 feet; and at the distance of the moon it is $16.1069 \cdot \sin^2 57'4''.17$ nearly. But in order to have this quantity more exactly it must be multiplied by $\frac{357}{358}$, because it is found by the theory of the moon's motion, that the action of the sun on the moon diminishes its gravity to the earth by a quantity, the constant part of which is equal to the 358th part of that gravity.

Again, it must be multiplied by $\frac{76}{75}$, because the moon in her relative motion round the earth, is urged by a force equal to the sum of the masses of the earth and moon divided by the square of *Cm*, their mutual distance. It appears by the theory of the tides that the mass of the

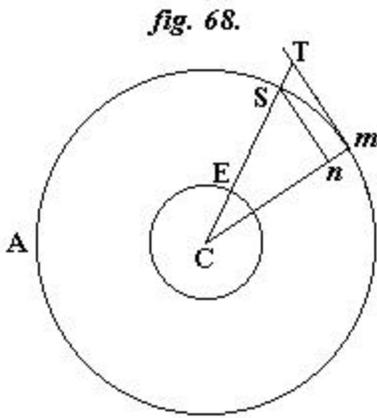
moon is only the $\frac{1}{75}$ of that of the earth which is taken as the unit of measure; hence the sum of the masses of the two bodies is

$$1 + \frac{1}{75} = \frac{76}{75}.$$

Then if the terrestrial attraction be really the force that retains the moon in her orbit, she must fall through

$$16.1069 \times \sin^2 57' 4'' .17 \times \frac{357}{358} \times \frac{76}{75} = 0.00448474$$

of a foot in a second.



333. Let mS , fig. 68, be the small arc which the moon would describe in her orbit in a second, and let C be the centre of the earth. If the attraction of the earth were suddenly to cease, the moon would go off in the tangent mT ; and at the second she would be in T instead of S ; hence the space that the attraction of the earth causes the moon to fall through in a second, is equal to mn the versed sine of the arc Sm .

The arc Sm is found by simple proportion, for the periodic time of the moon is 27.32166 days or 2,360,591'', and since the lunar orbit without sensible error may be assumed equal to the circumference of a circle whose radius is the mean distance of the moon from the earth; it is

$$O \quad \zeta \quad ' \quad ,$$

therefore¹⁴

$$2,360,591'' : 1'' :: 2Cm \cdot \frac{355}{113} : Sm$$

and¹⁵

$$Sm = \frac{2(355) \cdot Cm \cdot 1^{\text{sec.}}}{113(2,360,591^{\text{sec.}})}.$$

The arc Sm is so small that it may be taken for its chord, therefore $(mS)^2 = Cm \cdot mn$; hence¹⁶

$$\frac{4(355)^2(Cm)^2}{(113)^2(2,360,591'')^2} = 2Cm \cdot mn;$$

consequently

$$mn = \frac{2(355)^2 \cdot Cm}{(113)^2(2,360,591'')^2}.$$

Again, the radius CE of the earth in the latitude the square of whose sine is $\frac{1}{3}$, is computed to be 20,898,700 feet from the mensuration of the degrees of the meridian: and since

$$\frac{CE}{Cm} = \sin 57' 4''.17,$$

$$Cm = \frac{CE}{\sin 57' 4''.17} = \frac{20,898,700}{\sin 57' 4''.17},$$

consequently,

$$mn = \frac{2(355)^2(20,898,700)}{(113)^2(2,360,591'')^2 \sin 57' 4''.17} = 0.00445983$$

of a foot, which is the measure of the deflecting force at the moon. But the space described by a body in one second from the earth's attraction at the distance of the moon was shown to be 0.00448474 of a foot in a second; the difference is therefore only the 0.00002491 of a foot, a quantity so small, that it may safely be ascribed to errors in observation.

334. Hence it appears that the force¹⁷ that retains the moon in her orbit is terrestrial gravity, diminished in the ratio of the square of the distance. The same law then, which was proved to apply to a system of satellites, by a comparison of the squares of the times of their revolutions, with the cubes of their mean distances, has been demonstrated to apply equally to the moon, by comparing her motion with that of bodies falling at the surface of the earth.

335. In this demonstration, the distances were estimated from the centre of the earth, and since the attractive force of the earth is of the same nature with that of the other celestial bodies, it follows that the centre of gravity of the celestial bodies is the point from whence the distances must be estimated, in computing the effects of their attraction on substances at their surfaces, or on bodies in space.

336. Thus the sun possesses an attracting force, diminishing to infinity inversely as the squares of the distances, which includes all the bodies of the system in its action; and the planets which have satellites exert¹⁸ a similar influence over them.

Analogy would lead us to suppose that the same force exists in all the planets and comets; but that this is really the case will appear, by considering that it is a fixed law of nature that one body cannot act upon another without experiencing an equal and contrary reaction from that body: hence the planets and comets, being attracted towards the sun, must reciprocally attract the sun towards them according to the same law; for the same reason, satellites attract their planets. This property of attraction being common to planets, comets, and satellites, the gravitation of the heavenly bodies towards one another may be considered as a general principle of the¹⁹ universe; even the irregularities in the motions of these bodies are susceptible of being so well explained by this principle, that they concur in proving its existence.

337. Gravitation is proportional to the masses; for supposing the planets and comets to be at the same distance from the sun, and left to the action of gravity, they would fall through equal heights in equal times. The nearly circular orbits of the satellites prove that they gravitate like their planets towards the sun in the ratio of their masses: the smallest deviation from that ratio would be sensible in their motions, but none depending on that cause has been detected by observation.

338 Thus the planets, comets, and satellites, when at the same distance from the sun, gravitate as their masses; and as reaction is equal and contrary to action, they attract the sun in the same ratio; therefore their action on the sun is proportional to their masses divided by the square of their distances from his centre.

339. The same law obtains on earth; for very correct observations with the pendulum prove, that were it not for the resistance of the air, all bodies would fall towards its centre with the same velocity. Terrestrial bodies then gravitate towards the earth in the ratio of their masses, as the planets gravitate towards the sun, and the satellites towards their planets. This conformity of nature with itself upon the earth, and in the immensity of the heavens, shows, in a striking manner, that the gravitation we observe here on earth is only a particular case of a general law, extending throughout the system.

340. The attraction of the celestial bodies does not belong to their mass alone taken in its totality, but exists in each of their atoms, for if the sun acted on the centre of gravity of the earth without acting on each of its particles separately, the tides would be incomparably greater, and very different from what they now are. Thus the gravitation of the earth towards the sun is the sum of the gravitation of each of its particles; which in their turn attract the sun as their respective masses; besides, everything on earth gravitates towards the centre of the earth proportionally to its mass; the particle then reacts on the earth, and attracts it in the same ratio; were that not the case, and were any part of the earth however small not to attract the other part as it is itself attracted, the centre of gravity of the earth would be moved in space in virtue of this gravitation, which is impossible.

341. It appears then, that the celestial phenomena when compared with the laws of motion, lead to this great principle of nature, that all the particles of matter mutually attract each other as their masses directly, and as the squares of their distances inversely.

342. From the universal principle of gravitation, it may be foreseen, that the comets and planets will disturb each other's motion, so that their orbits will deviate a little from perfect ellipses; and the areas will no longer be exactly proportional to the time: that the satellites, troubled in their paths by their mutual attraction, and by that of the sun, will sensibly deviate from elliptical motion: that the particles of each celestial body, united by their mutual attraction, must form a mass nearly spherical; and that the resultant of their action at the surface of the body, ought to produce there all the phenomena of gravitation. It appears also, that centrifugal force arising from the rotation of the celestial bodies must alter their spherical form a little by flattening them at their poles; and that the resulting force of their mutual attractions not passing through their centres of gravity, will produce those motions that are observed in their axes of rotation. Lastly, it is clear that the particles of the ocean being unequally attracted by the sun and

moon, and with a different intensity from the nucleus of the earth, must produce the ebb and flow of the sea.

343. Having thus proved from Kepler's laws, that the celestial bodies attract each other directly as their masses, and inversely as the square of the distance, Laplace inverts the problem, and assuming the law of gravitation to be that of nature, he determines the motions of the planets by the general theorem in article 144, and compares the results with observation.

Notes

¹ This word is not capitalized in 1st edition.

² The 1st edition omits the word "half" (published erratum).

³ The equation reads $F = \frac{c}{a(1-e^2)} \cdot \frac{1}{r^2}$ in the 1st edition (published erratum).

⁴ The expression reads $F = \frac{c}{a(1-e^2)} \cdot \frac{1}{r^2}$ in the 1st edition (published erratum).

⁵ Punctuation in this expression in the 1st edition is misplaced as $F = \frac{4p^2}{k^2} \cdot \frac{1}{r^2}; = h \cdot \frac{1}{r^2}$.

⁶ This reads, "The three laws of Kepler," for "The discovery of the laws of Kepler" in the 1st edition (published erratum).

⁷ See note 6, *Book II, Chapter I.*

⁸ This reads $a' \frac{2p^2}{T_i^2}$ in the 1st edition.

⁹ This reads $\text{sine } CmE = \frac{R}{r} \sin z$ in the 1st edition.

¹⁰ The punctuation in 1st edition is contained within the parenthesis as $p + p' = P \{ \sin z + \sin z' \}$

¹¹ This reads ZE, Z'E' for zE, z'E' in the 1st edition (published erratum).

¹² Somerville spells the name La Caille in the 1st edition.

¹³ Lacaille, Nicolas Louis de, 1713-1762 astronomer, born in Rumigny, France. From 1750 to 1754 he compiled a catalogue of nearly 10,000 southern stars and was first to measure the arc of the meridian in South Africa. Lacaille's *Coelum Australe Stelliferum* was published in 1763.

¹⁴ Expressed $2360591'' : 1' :: 2Cm \frac{355}{113} : Sm$ in the 1st edition.

¹⁵ Expressed $Sm = \frac{2(355) \cdot Cm \cdot 1^{\text{sec.}}}{113(2360591)}$ in the 1st edition.

¹⁶ Expressed $\frac{4(355)^2 (Cm)^2}{(113)^2 (2360591'')^2} = 2Cm \cdot mm$ in the 1st edition.

¹⁷ This reads "principal force" for "force." in the 1st edition (published erratum).

¹⁸ This reads "exact" for "exert" in the 1st edition (published erratum).

¹⁹ This reads "this" for "the" in the 1st edition (published erratum).