BOOK II

CHAPTER IV

ON THE ELLIPTICAL MOTION OF THE PLANETS

359. THE elliptical orbit of the earth is the plane of the ecliptic: the plane of the terrestrial equator cuts the plane of the ecliptic in a line passing through the vernal and autumnal equinoxes.

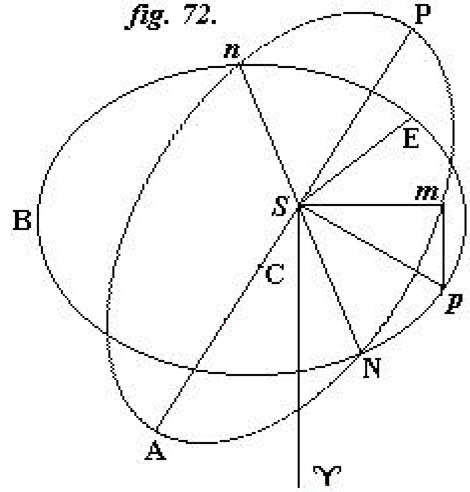
The vernal equinox is assumed as an origin from whence the angular distances of the heavenly bodies are estimated. Astronomers designate that point by the character ♈ the first point of Aries, although these points have not coincided for 2230 years, on account of the precession or retrograde motion of the equinoxes.

360. Angular distance from the vernal equinox, or first point of Aries, estimated on the plane of the ecliptic, is longitude, which is reckoned from west to east, the direction in which the bodies of the solar system revolve round the sun. For example, let E*n*BN, fig. 72, represent the ecliptic, *S* the sun, and ♈ the first point of Aries, or vernal equinox. If the earth be in E, its longitude is the angle ♈*S*E.

361. The earth alone moves in the plane of the ecliptic, the orbits of the other bodies of the system are inclined to it at small angles; so that the planets, in their revolutions, are sometimes seen above that plane, and sometimes below it. The angular distance of a planet above or below the plane of the ecliptic, is its latitude; when the planet is above that plane, it is said to have north latitude, and when below it, south latitude. Latitude is reckoned from zero to 180°.

362. Let E*n*BN represent the plane of the ecliptic, and let *m* be a planet moving round the sun *S* in the direction *m*P*n*, the orbit being inclined to the ecliptic at the angle PNE; the part of the orbit NP*n* is supposed to be above the plane of the ecliptic, and NA*n* below it. The line N*Sn*, which is the intersection of the plane of the orbit with the plane of the ecliptic, is the line of nodes; it always passes through the centre of the sun. When the planet is in N, it is in its ascending node; when in *n*, it is in its descending node. Let *mp* be a perpendicular from *m* on the plane of the ecliptic, *Sp* is the projection of the radius vector *Sm*, and is the curtate[1] distance of the planet from the sun. ♈*Sm* is the longitude of the ascending node; and it is clear that the longitude of *n*, the descending node, is 180° greater. The longitude of *m* is ♈*Sm*, or ♈*Sp*, according as it is estimated on the orbit, or on the ecliptic; and *mSp*, the angular height *m* above the plane of the ecliptic, is its latitude. As the position of the first point of Aries is known, it is evident that the place of a planet *m* in its orbit is found, when the angles ♈*Sm*, *mSp*, and *Sm*, its

fig. 72.



distance from the sun, are known at any given time, or ♈*Sp*, *pSm*, and *Sp*, which are more generally employed. But in order to ascertain the real place of a body, it is also requisite to know the nature of the orbit in which it moves, and the position of the orbit in space. This depends on six constant quantities, AP, the greater axis of the ellipse; $\frac{\mathrm{C}S}{\mathrm{CP}}$, the eccentricity; ♈*Sp*, the longitude of P, the perihelion; ♈*S*N, the longitude of N, the ascending node; ENP, the inclination of the orbit on the plane of the ecliptic; and on the longitude of the epoch, or position of the body at the origin of the time.

These six quantities, called the elements of the orbit, are determined by observation; therefore the object of analysis is to form equations between the longitude, latitude, and distance from the sun, in values of the time; and from them to compute tables which will give values of these three quantities, corresponding to any assumed time, for a planet or satellite; so that the situation of every body in the system may be ascertained by inspection alone, for any time past, present, or future.

363. The motion of the earth differs from that of any other planet, only in having no latitude, since it moves in the plane of the ecliptic, which passes through the centre of the sun. In consequence of the mutual attraction of the celestial bodies, the position of the ecliptic is variable to a very minute extent; but as the variation is known, its position can be ascertained.

364. The motions of the celestial bodies, and the positions of their orbits, will be referred to the known position of this plane at some assumed epoch, say 1750, unless the contrary be expressly mentioned. It will therefore be assumed to be the plane of the co-ordinates x and y, and will be called the FIXED PLANE.

Motion of one Body

365. If the undisturbed elliptical motion of one body round the sun be considered, the equations in article 346[2] become

$$\begin{aligned}\frac{d^2x}{dt^2}+\frac{mx}{r^3}&=0,\\ \frac{d^2y}{dt^2}+\frac{my}{r^3}&=0,\\ \frac{d^2z}{dt^2}+\frac{mz}{r^3}&=0,\end{aligned}\qquad \textbf{(89)}$$

where m is put for $S+m$, the sum of the masses of the sun and planet, and $r=\sqrt{x^2+y^2+z^2}$.

In these three equations, the force is inversely as the square of the distance; they ought therefore to give all the circumstances of elliptical motion. Their finite values will give x, y, z, in values of the time, which may be assumed at pleasure: thus the place of the body in its elliptical orbit will be known at any instant; and as the equations are of the second order, six arbitrary constant quantities will be introduced by their integration, which determine the six elements of the orbit.

366. These give the motion of the planet with regard to the sun; but the equations

$$0=\frac{d^2\bar{x}}{dt^2}-\frac{mx}{r^3};\quad 0=\frac{d^2\bar{y}}{dt^2}-\frac{my}{r^3};\quad 0=\frac{d^2\bar{z}}{dt^2}-\frac{mz}{r^3},$$

of article 346, give values of $\bar{x}$, $\bar{y}$, $\bar{z}$, in terms of the time which will determine the motion of the sun in space; for if the first of them be multiplied by $S+m$, and added to

$$\frac{d^2x}{dt^2}+\frac{(S+m)\,x}{r^3}=0,$$

multiplied by m, their sum will be

$$(S+m)\frac{d^2\bar{x}}{dt^2}+m\frac{d^2x}{dt^2}=0,$$

the integral of which is

$$\bar{x}=a+bt-\frac{mx}{S+m};$$

in the same manner,

$$\bar{y}=a'+b't-\frac{my}{S+m},$$

$$\bar{z}=a''+b''t-\frac{mz}{S+m}.$$

These equations give the motion of the sun in space accompanied by m; and as they are the same for each body, if $\sum m$ be substituted for m, they will determine the absolute motion of the sun attended by the whole system, when the relative motions of m, m', m'', &c., are known.

367. But in order to ascertain the values of x, y, z, the equations (89) must be integrated. Since these equations are linear and of the second order, their integrals must contain six constant quantities. They are also symmetrical and so connected, that any one of the variable quantities x, y, z, depends on the other two. M. Pontécoulant[3] has determined these integrals with great elegance and simplicity in the following manner.

368. If the first of the equations (89) of elliptical motion multiplied by y, be subtracted from the second multiplied by x, the result will be

$$\frac{xd^2y-yd^2x}{dt^2}=0;$$

consequently,

$$\frac{xdy-ydx}{dt}=c.$$

In the same way it is easy to find that

$$\frac{zdx - xdz}{dt} = c'; \quad \frac{ydz - zdy}{dt} = c'',$$

where c, c', c'', are arbitrary constant quantities introduced by integration. Again, if the first of the same equations be multiplied by $2dx$, the second by $2dy$, and the third by $2dz$, their sum will be

$$\frac{2dxd^2x + 2dyd^2y + 2dzd^2z}{dt^2} + \frac{2m\left(xdx + ydy + zdz\right)}{r^3} = 0.$$

But

$$r^2 = x^2 + y^2 + z^2;$$

whence

$$rdr = xdx + ydy + zdz;$$

and the integral of the preceding equation is

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} - \frac{2m}{r} + \frac{m}{a} = 0, \tag{90}$$

$\frac{m}{a}$ being an arbitrary constant quantity. If

$$\frac{d^2y}{dt^2} = -\frac{my}{r^3}, \text{ multiplied by } c'' = \frac{ydz - zdy}{dt},$$

be subtracted from

$$\frac{d^2x}{dt^2} = -\frac{mx}{r^3}, \text{ multiplied by } c' = \frac{zdx - xdz}{dt},$$

the result will be

$$\frac{c'd^2x - c''d^2y}{dt} = \frac{mx}{r^3}\left(xdz - zdx\right) - \frac{my}{r^3}\left(zdy - ydz\right)$$

$$= \frac{m\left(rdz - zdr\right)}{r^2} = md \cdot \frac{z}{r}.$$

Whence

$$f + \frac{mz}{r} = \frac{c'dx - c''dy}{dt};$$

and by a similar process values of[4]

$$m\,.\,d\,\frac{y}{r},\ \text{and}\ m\,.\,d\,\frac{x}{r},$$

may be found, the integrals of which are

$$f' + \frac{my}{r} = \frac{c''dz - cdx}{dt}; \quad f'' + \frac{mx}{r} = \frac{cdy - c'dz}{dt}.$$

369. Thus the integrals of equations (89) are,

$$\begin{gathered}
c = \frac{xdy - ydx}{dt}; \quad c' = \frac{zdx - xdz}{dt}; \quad c'' = \frac{ydz - zdy}{dt}; \\
f + \frac{mz}{r} = \frac{c'dx - c''dy}{dt}, \\
f' + \frac{my}{r} = \frac{c''dz - cdx}{dt}, \\
f'' + \frac{mx}{r} = \frac{cdy - c'dz}{dt}, \\
\frac{m}{a} - \frac{2m}{r} + \frac{dx^2 + dy^2 + dz^2}{dt^2} = 0,
\end{gathered} \tag{91}$$

containing the seven arbitrary constant quantities c, c', c'', f, f', f'', and a.

370. As two equations of condition exist among the constant quantities, they are reduced to five that are independent, consequently two of the seven integrals are included in the other five. For if the first of these equations be multiplied by z, the second by y, and the third by x, their sum is

$$cz + c'y + c''x = 0 \tag{92}$$

Again, if the fourth integral multiplied by c, be added to the fifth multiplied by c',

$$fc + f'c' + m\frac{cz + c'y}{r} = c'' \cdot \frac{c'dz - cdy}{dt};$$

but

$$cz + c'y = -c''x;$$

hence

$$-\frac{fc + f'c'}{c''} + m\frac{x}{r} = \frac{cdy - c'dz}{dt};$$

but this coincides with the sixth integral, when

$$f'' = -\frac{fc + f'c'}{c''}, \text{ or } f''c'' + f'c' + fc = 0 \,.$$

The six arbitrary quantities being connected by this equation of condition, the sixth integral results from the five preceding.

If the squares of f, f', and f'', from the fourth, fifth, and sixth integrals be added, and [letting] $f^2 + f'^2 + f''^2 = l^2$, they give

$$l^2 - m^2 = \left(c^2 + c'^2 + c''^2\right)\left\{\frac{dx^2 + dy^2 + dz^2}{dt^2} - \frac{2m}{r}\right\} - \left\{\frac{cdz + c'dy + c''x}{dt}\right\}^2$$

but

$$cz + c'y + c''x = 0; \text{ hence } cdz + c'dy + c''dx = 0\,;$$

consequently, if

$$c^2 + c'^2 + c''^2 = h^2\,,$$

[then]

$$0 = \frac{dx^2 + dy^2 + dz^2}{dt^2} - \frac{2m}{r} + \frac{m^2 - l^2}{h^2}\,,$$

and comparing this equation with the last of the integrals in article 369, it will appear that

$$\frac{m^2 - l^2}{h^2} = \frac{m}{a}\,;$$

thus, the last integral is contained in the others; so that the seven integrals and the seven constant quantities are in reality only equal to five distinct integrals and five constant quantities.

371. Although these are insufficient to determine x, y, z, in functions of the time, they give the curve in which the body m moves. For the equation

$$cz + c'y + c''x = 0$$

is that of a plane passing through the origin of the co-ordinates, whose position depends on the constant quantities c, c', c''. Thus the curve in which m moves is in one plane. Again, if the fourth of the integrals in article[5] 369 be multiplied by z, the fifth by y, and the sixth by x, their sum will be

$$fz + f'y + f''x + \frac{m\left(x^2 + y^2 + z^2\right)}{r} = c\frac{\left(xdy - ydx\right)}{dt} + c'\frac{\left(zdx - xdz\right)}{dt} + c''\frac{\left(ydz - zdy\right)}{dt}\,;$$

but in consequence of the three first integrals in article 369, it becomes[6]

$$0 = mr - \left(c^2 + c'^2 + c''^2\right) + fz + f'y + f''x\,,$$

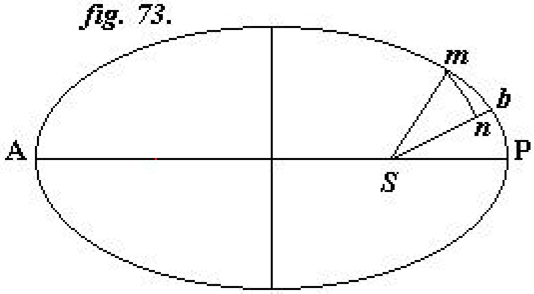
or

$$0 = mr - h^2 + fz + f'y + f''x.$$

This equation combined with

$$cz + c'y + c''x = 0, \quad \text{and} \quad r^2 = x^2 + y^2 + z^2,$$

fig. 73.



gives the equation of conic sections, the origin of r being in the focus.

372. Thus the planets and comets move in conic sections having the sun in one of their foci, and their radii vectores describe areas proportional to the time; for if dv represent the indefinitely small arc mb, fig. 73, contained between

$$Sm = r \text{ and } Sb = r + dr,$$

then

$$(mb)^2 = dx^2 + dy^2 + dz^2 = r^2dv^2 + dr^2,$$

but the sum of the squares of the three first of equations (91) is

$$\frac{(x^2 + y^2 + z^2)(dx^2 + dy^2 + dz^2)}{dt^2} - \frac{(xdx + ydy + zdz)}{dt^2} = h^2,$$

or

$$\frac{r^2(dx^2 + dy^2 + dz^2)}{dt^2} - \frac{r^2dr^2}{dt^2} = h^2;$$

hence

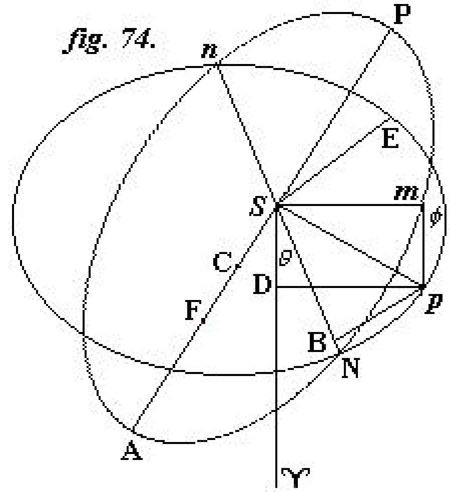
$$dv = \frac{hdt}{r^2}. \qquad \textbf{(93)}$$

373. Thus the area $\frac{1}{2}r^2dv$ described by the radius vector r or Sm is proportional to the time dt, consequently the finite area described in a finite time is proportional to the time. It is evident also, that the angular motion of m round S is in each point of the orbit, inversely as the square of the radius vector, and as very small intervals of time may be taken instead of the indefinitely small instants dt, without sensible error, the preceding equation will give the horary[7] motion of the planets and comets in the different points of their orbits.

Determination of the Elements of Elliptical Motion

374. The elements of the orbit in which the body m moves depend on the constant quantities c, c', c'', f, f', f'', and $\frac{m}{a}$. In order to determine them, it must be observed that in the equations (89) the co-ordinates x, y, z, are SB, Bp, pm, fig. 74; but if they be referred to[8] ϒS the line of the equinoxes, so that $S\mathrm{D} = x'$, $\mathrm{D}p = y'$, $pm = z'$, and if ϒSN, ENP, the longitude of the node and inclination of the orbit on the fixed plane be represented by q and f ; it is evident, from the method of the co-ordinates in article 225, that

fig. 74.



$$x' = x\cos q + y\sin q,$$
$$y' = y\cos q - x\sin q,$$
$$z' = y\tan f,$$

consequently

$$z = y\cos q\ \tan f - x\sin q\ \tan f\ ;$$

but if this be compared with

$$0 = c''x + c'y + cz\ ,$$

it will be found that

$$c' = -c\cos q\ \tan f\ ,$$
$$c'' = c\sin q\ \tan f\ ,$$

whence

$$\left\{\begin{array}{l} \tan q = -\dfrac{c''}{c'} \\ \tan f = \dfrac{\sqrt{c'^2 + c''^2}}{c} \end{array}\right\} \qquad \textbf{(94)}$$

Thus this position of the nodes and the inclination of the orbit are given in terms of the constant quantities c, c', c''.

375. Now $r^2 = x^2 + y^2 + z^2$, and $rdr = xdx + ydy + zdz$, but at the perihelion the radius vector r is a minimum; hence $dr = 0$, therefore

$$xdx + ydy + zdz = 0\,.$$

Let $x_{/}$, $y_{/}$, $z_{/}$, be the co-ordinates of the planet when in perihelio, then, substituting the values of c, c', c'', from [article] 369[9] in the equations in f' and f'' of the same number, and dividing the one by the other, the result in consequence of the preceding relation will be

$$\frac{y_{/}}{x_{/}} = \frac{f'}{f''}\,.$$

But if $v_{/}$ be the angle ♈*S*E, the projection of the longitude of the perihelion on the plane N*pn*, then $\frac{y_{/}}{x_{/}} = \tan v_{/}$; hence

$$\tan v_{/} = \frac{f'}{f''};$$

which determines the position of the greater axis of the conic section.

If $\frac{dx^2 + dy^2 + dz^2}{dt^2}$ be eliminated from the equation

$$r^2\left(\frac{dx^2 + dy^2 + dz^2}{dt^2}\right) - \frac{r^2 dr^2}{dt^2} = h^2,$$

by means of the last (91) the result will be

$$2mr - \frac{mr^2}{a} - \frac{r^2 dr^2}{dt^2} = h^2;$$

but at the extremities of the greater axis $dr = 0$, because the radius vector is either a maximum or minimum at these points, therefore at the aphelion and perihelion

$$0 = r^2 - 2ar + \frac{ah^2}{m};$$

whence

$$r = a \pm a\sqrt{1 - \frac{h^2}{ma}}.$$

The sum of these two values of *r* is the major axis of the conic section, and their difference is F*S* or double the eccentricity.

376. Thus *a* is half of AP, fig. 75, the major axis of the orbit, or it is the mean distance of *m* from *S*; and $\sqrt{1 - \frac{h^2}{ma}}$, is the ratio of the eccentricity to half the major axis. Let this ratio be represented by *e*, then as it was shown that $\frac{m^2 - l^2}{h^2} = \frac{m}{a}$; so also

$$me = l = \sqrt{f^2 + f'^2 + f''^2}.$$

Thus all the elements that determine the nature of the conic section and its position in space are known.

377. The three equations

$$r^2 = x^2 + y^2 + z^2, \quad mr - h^2 + fz + f'y + f''z = 0, \text{ and } c''x + c'y + cz = 0,$$

give x, y, z, in functions of r; but in order to have values of these co-ordinates in terms of the time, r must be found in terms of the same, which requires another integration. Resume the equation

$$2mr - \frac{mr^2}{a} - \frac{r^2 dr^2}{dt^2} = h^2,$$

then

$$\sqrt{1 - \frac{h^2}{ma}} = e,$$

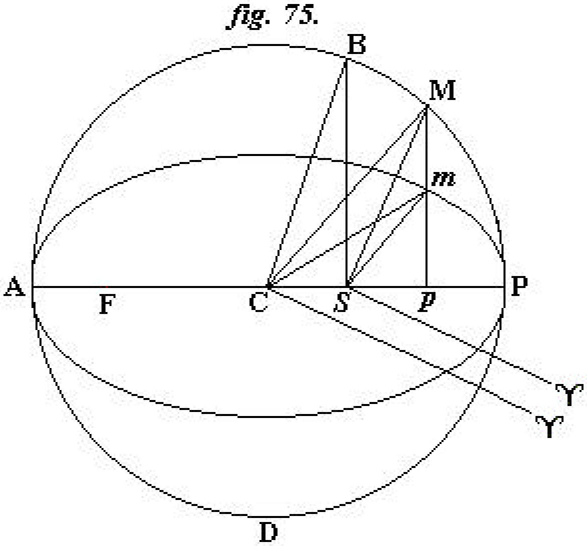
gives

$$h^2 = am\left(1 - e^2\right)$$

therefore

$$dt = \frac{rdr}{\sqrt{m}\sqrt{2r - \frac{r^2}{a} - a\left(1 - e^2\right)}}.$$

fig. 75.



To integrate this equation, a value of r must be found from the conic sections. Let AmP, fig. 75, be an ellipse whose major axis is $2a$, its minor axis $2b$, the eccentricity $CS = e'$, and the radius vector $Sm = r$.

Let the circle PMA be described on the major axis, draw the perpendicular Mp through m, and join SM, CM, and Cm. Then

$$r^2 = Sp^2 + pm^2, \text{ and if MCP}=u,$$
$$Sp = Cp - CS = a\cos u - e',$$

or making

$$e = \frac{e'}{a}, \quad Sp^2 = a^2\left(\cos u - e\right)^2.$$

Again,

$$pm^2 = b^2 . \sin^2 u = b^2\left(1 - \cos^2 u\right);$$

but

$$b^2 = a^2 - e'^2 = a^2\left(1 - e^2\right);$$

hence

$$r^2 = a^2\left(1 - e^2\right)\left(1 - \cos^2 u\right) + a^2\left(\cos u - e\right)^2,$$

and

$$r = a\{1 - e\cos u\}.$$

This value of r and its differential being substituted in the value of dt it becomes

$$dt = \frac{a^{\frac{3}{2}}}{\sqrt{u}} \cdot du\left(1 - e\cos u\right)$$

the integral of which is

$$t + k = \frac{a^{\frac{3}{2}}}{\sqrt{u}}\{u - e\sin u\}; \qquad \textbf{(95)}$$

k being an arbitrary constant quantity.

This equation gives u and consequently r in terms of t, and as x, y, z, are given in functions of r, the values of these co-ordinates are known at any instant.

When $u = 1$ the values of dt and h^2 become

$$\frac{rdr}{\sqrt{2r - \frac{r^2}{a} - a\left(1 - e^2\right)}}, \text{ and } a\left(1 - e^2\right),$$

and when substituted in $dv = \frac{hdt}{r^2}$, the result is

$$dv = \frac{dr \cdot \sqrt{a\left(1 - e^2\right)}}{r\sqrt{2r - \frac{r^2}{a} - a\left(1 - e^2\right)}},$$

or

$$dv = \frac{\frac{a\left(1 - e^2\right)}{e} d \cdot \frac{1}{r}}{\sqrt{1 - \left\{\frac{a\left(1 - e^2\right)\frac{1}{r} - 1}{e}\right\}}};$$

the integral of which is

$$v = v\ + arc\left\{\cos = \frac{a\left(1 - e^2\right)\frac{1}{r} - 1}{e}\right\},$$

reciprocally[10]

$$r = \frac{a\left(1 - e^2\right)}{1 + e\cos\left(v - v\,\right)},$$

which is the general equation to the conic sections, when the origin of r the radius vector is in the focus; a is half the greater axis, and

$$\cos\left(v - v\right) = \cos\ (\text{♈}Sm - \text{♈}SP),\ \text{fig. 77.}$$

Elements of the Orbit

378. Thus the finite values of the equations of elliptical motion are completely determined,

Six arbitrary constant quantities have been introduced, namely,

2a, the greater axis of the orbit.
e, the ratio of the eccentricity to half the greater axis.
$v_{/}$, the projection of the longitude of the perihelion.
q, the longitude of the ascending mode.
f, the inclination of the orbit on the plane of the ecliptic, and
$\in$, the longitude of the epoch.

The two first determine the nature of the orbit, the three following its position in space, and the last is relative to the position of the body at a given epoch; or, which is the same thing, it depends on the instant of its passage at the perihelion.

Equations of Elliptical Motion

379. It now becomes necessary to determine three equations which will give values of the longitude and latitude ♈*Sm*, *mSp*, and the distance *Sm*, fig. 72, in terms of the time from whence tables of the elliptical motions of the planets and satellites may be computed.

380. The motion of a body in an ellipse is not uniform, its velocity is greatest at the perihelion, and least at the aphelion, varying with the angle P*Sm*, which is the true angular motion of the planet; but if the circle PBAD, fig. 75, be described from the centre of the ellipse with the semigreater axis CP, or mean distance from S as radius, the motion of the planet in this circle would be uniform. This is called the mean motion of a body.

381. Were the motion of a planet uniform, the angle P*Sm* described by the planet in any interval of time after leaving perihelion might be found by simple proportion from knowing the periodic time, or time in which it describes 360°; but in order to preserve the equable description of areas, the true place of the planet will be before the mean place in going from perihelion to aphelion; and from aphelion to perihelion the true place will be behind the mean place. These angles are estimated from west to east, the direction in which the bodies of the system move, beginning at the perihelion. If, however, they are estimated from the aphelion, it is only necessary to add 180° to each.

382. The angular distance PCB between the perihelion and the mean place, is the mean anomaly, *PSm* the angular distance between the true place and the perihelion is the true anomaly; and *mS*B the angle at the sun, contained between the true and the mean place is called the equation of the centre. If then the mean anomaly be increased or diminished by the equation of the centre, the result will be the true place of the planet in its orbit. The equation of the centre is zero, both at the perihelion and aphelion, for if these points the true and mean places of the planets coincide; it is greatest when the planet is in quadratures, and at its maximum it is equal to an angle measured by twice the eccentricity of the orbit.

383. The mean place of a planet, at any given time may be found by simple proportion from its periodic time. The true place of the planet in its orbit, and its distance from the sun, may be found in terms of its mean place by help of the angle PCM, called the eccentric anomaly.

If the time be estimated from the perihelion,[11] $k=0$, which reduces equation (95) to[12]

$$t=\frac{a^{\frac{3}{2}}}{\sqrt{u}}\cdot\left(u-e.\sin u\right),\ \text{or}\ nt=u-e\sin u,\ \text{if}\ n=\frac{\sqrt{u}}{a^{\frac{3}{2}}}.$$

If the angles u and v be estimated from the perihelion, a comparison of the values of r in article 377 gives

$$1-e\cos u=\frac{1-e^2}{1+e\cos v},$$

whence

$$\cos v=\frac{\cos u-e}{1-e\cos u};\quad \sin v=\frac{\sin u\cdot\sqrt{1-e^2}}{1-e\cos u},$$

therefore

$$\tan\tfrac{1}{2}v=\sqrt{\frac{1+e}{1-e}}\cdot\tan\tfrac{1}{2}u.$$

384. The motions of the celestial bodies in elliptical orbits are therefore obtained from the three equations

$$\begin{gathered}nt=u-e\sin u,\\ r=a\left(1-e\cos u\right)\\ \tan\tfrac{1}{2}v=\sqrt{\frac{1+e}{1-e}}\cdot\tan\tfrac{1}{2}u.\end{gathered}\qquad\textbf{(96)}$$

Where
nt = PCB = mean anomaly, fig. 75,
v = P*Sm* = true anomaly,
u = PCM = eccentric anomaly,
r = *Sm* = radius vector,
a = CP = mean distance, and

$$e = \frac{\mathrm{C}S}{\mathrm{CP}} = \text{the ratio of the eccentricity to the mean distance}.$$

385. It appears from these expressions that when u becomes $u + 360^\circ$, r remains the same; and as v is then augmented by 360°, the planet returns to the same point of its orbit, having moved through four right angles, and the time becomes[13] $\mathrm{T} = \frac{a^{\frac{3}{2}}}{\sqrt{u}} \cdot 360^\circ$; so that the time of a complete revolution is independent of the eccentricity, and only depends on $2a$, the major axis of the orbit; it is consequently the same as if the planet described a circle at its mean distance from the sun; for in this case $e = 0$, $r = a$, $u = nt$, $v = u$, consequently $v = nt$; the arcs described are therefore proportional to the time, and the planet moves uniformly in the circle whose radius is a. Generally nt represents the arc that a body would describe in the time t, if it set out from the perihelion at the same instant with a planet m, and moved with a uniform velocity represented by n in a circle described on the major axis of the orbit as diameter. This body would pass the perihelion and aphelion at the same instant with the planet m, but in one half of its revolution the planet would precede the body, and in the other half it would fall behind it. If $a = 1$, $m = 1$, then $n = 1$, and $v = t$, the time will therefore be expressed by the arcs described by the planet in the circle whose radius is unity.

Astronomers generally compare the motions of the solar system with those of the earth; they take the mean distance of the sun from the earth as the unit of distance, the sum of the masses of the sun and earth as the unit of mass; and supposing the time to be estimated in mean solar days, the unit of time will be represented by the arc that the earth describes round the sun in one day with its mean motion.

Determination of the Eccentric Anomaly in functions of the Mean Anomaly

386. If a value of u could be found in terms of nt from the first of these equations, both r and v, and consequently the place of the planet in its orbit at any instant, would be known from the two last.

Now an arc and its sine are incommensurate quantities, so that the one can only be obtained in functions of the other by an infinite series. Therefore a value of u in terms of nt must be found by an infinite series from the first of the preceding equations; but unless the terms of the series decrease rapidly in value u cannot be obtained, for a few of the first terms being computed, the value of the remaining part of the series must be so small that it may be neglected without sensible error. The small eccentricities of the orbits of the planets and satellites afford the means of approximation, for e the ratio of the eccentricity to half the greater axis is still smaller, consequently the powers of such quantities decrease rapidly, and therefore the second part of the equation $u = nt + e\sin u$ may be expanded into a series in functions of the time, and according to the powers of e, which will be sufficiently convergent. This may be accomplished by Maclaurin's *Theorem*,[14] for if u' be the value of u when $e = 0$,

$$u = u' + e \cdot \frac{du'}{de} + \frac{e^2}{1.2} \cdot \frac{d^2u'}{de^2} + \frac{e^3}{1.2.3} \cdot \frac{d^3u'}{de^3} + \text{\&c.}$$

But when $e=0$, $u=nt+e\sin u$, becomes $u'=nt$; and from the same equation

$$\frac{du}{de}=\frac{\sin u}{1-e\cos u};$$

or when

$$e=0,\quad \frac{du'}{de}=\sin nt\,.$$

Again,

$$\frac{d^2u}{de^2}=\frac{2\cos u\sin u}{\left(1-e\cos u\right)^2}-\frac{e\sin^3 u}{\left(1-e\cos u\right)^3};$$

or if[15]

$$e=0,\quad \frac{d^2u'}{de^2}=2\cos nt\sin nt$$

in the same manner, when $e=0$,

$$\frac{d^3u'}{de^3}=6\sin nt\cos^2 nt-3\sin^3 nt\,,$$

or

$$\frac{d^3u'}{de^3}=6\sin nt-9\sin^3 nt, \text{ \&c. \&c.}$$

But

$$2\cos nt\sin nt=\sin 2nt\,,$$

and

$$6\sin nt-9\sin^3 nt=-\tfrac{3}{4}\sin 3nt+\tfrac{9}{4}\sin 3nt\,;$$

hence

$$\frac{du'}{de}=\sin nt;\quad \frac{d^2u'}{de^2}=\frac{1}{2}\sin 2nt;\quad \frac{d^3u'}{de^3}=\frac{1}{2^2}\cdot\left\{3^2\sin 3nt-3\sin nt\right\}\ \text{\&c.}$$

consequently,[16]

$$\begin{aligned}u=nt+e\sin nt+\frac{e^2}{1.2.2}\cdot 2\sin 2nt+\frac{e^3}{2.3.2^2}\cdot\left\{3^2\sin 3nt-3\sin nt\right\}\\ +\frac{e^4}{2.3.4.2^3}\cdot\left\{4^3\sin 4nt-4.2^3\sin 2nt\right\}\\ +\frac{e^5}{2.3.4.5.2^4}\cdot\left\{5^4\sin 5nt-5.3^2\sin 3nt+\frac{5.4}{1.2}\sin nt\right\}\\ +\text{ \&c. \&c. \&c.}\end{aligned}$$

This series converges rapidly in most of the planetary orbits on account of the small value of the fraction which *e* expresses.

387. Having thus determined u for any instant, corresponding values of v and r may be obtained from the equations $r = a(1 - e\cos u)$ and

$$\tan \tfrac{1}{2} v = \sqrt{\frac{1+e}{1-e}} \cdot \tan \tfrac{1}{2} u ;$$

but it is better to expand these also into series ascending according to the powers of e; and in functions of the sines or cosines of the mean anomaly.

Determination of the Radius Vector in functions of the Mean Anomaly

Let r' be the value of r when $e = 0$, then[17]

$$r = r' + e\frac{dr'}{de} + \frac{e^2}{1.2} \cdot \frac{d^2 r'}{de^2} + \&\text{c}.$$

but as r is a function of e by the equation $r = a(1 - e\cos u)$; and u is a function of e by $u = nt + e\sin u$, therefore,[18]

$$\frac{dr}{de} = \frac{dr'}{de} + \frac{dr'}{du} \cdot \frac{du}{de}.$$

Now when $e = 0$, $\frac{r}{a} = 1$; and[19] $u = nt$. But the differentials of the same equations, when $e = 0$, are[20]

$$\frac{dr}{de} = -a\cos nt; \text{ and } \frac{du}{de} = \sin nt ;$$

consequently,

$$\frac{dr'}{de} = -a\cos nt + \sin nt \cdot \frac{dr}{ndt}, \text{ for } du = ndt ;$$

or it may be written,

$$\frac{dr'}{de} = -a\cos nt + \frac{d\int \sin nt \,.\, dr}{ndt}.$$

Again,

$$\frac{d^2 r'}{de^2} = \frac{d^2 \int \sin nt \,.\, dr}{ndt \cdot de};$$

but if $\int \sin nt \cdot dr$ be put for r in

$$\frac{dr}{de} = \frac{d\int \sin nt \,.\, dr}{ndt},$$

then,

$$\frac{d\int \sin nt \,.\, dr}{de} = \frac{d\int \sin^2 nt \,.\, dr}{ndt}.$$

And if this be substituted in the value of $\frac{d^2 r'}{de^2}$ it becomes[21]

$$\frac{d^2 r'}{de^2} = \frac{d^2 \int \sin^2 nt \,.\, dr}{(ndt)^2} = \frac{d \cdot \left(\sin^2 nt \cdot \frac{dr}{ndt} \right)}{ndt}.$$

The differential of the latter expression according to e is

$$\frac{d^3 r'}{de^3} = \frac{d^3 \int \sin^2 nt \,.\, dr}{(ndt)^2 \cdot de};$$

and making the same substitution, it becomes[22]

$$\frac{d^3 r'}{de^3} = \frac{d^3 \int \sin^3 nt \,.\, dr}{(ndt)^3} = \frac{d^2 \cdot \left(\sin^3 nt \cdot \frac{dr}{ndt} \right)}{(ndt)^2},$$

and so on. These coefficients being substituted,

$$r = a - ae \cos nt + e \sin nt \cdot \frac{dr}{ndt} + \frac{e^2}{2} \cdot \frac{d\left(\sin^2 nt \cdot \frac{dr}{ndt} \right)}{ndt} + \&c.$$

But

$$r = a(1 - e \cos nt) \text{ gives } \frac{dr}{ndt} = ae \,.\, \sin nt;$$

hence[23]

$$\frac{r}{a} = 1 - e \cos nt + e^2 \sin^2 nt + \frac{e^3}{2} \cdot \frac{d \cdot \sin^3 nt}{ndt} + \frac{e^4}{2.3} \cdot \frac{d^2 \cdot \sin^4 nt}{n^2 dt^2} + \&\text{c.}$$

Now

$$\sin^2 nt = \tfrac{1}{2} - \tfrac{1}{2} \cos 2nt,$$

$$\frac{d \cdot \sin^3 nt}{ndt} = 3 \sin^2 nt \cos nt = \tfrac{3}{4}\{\cos nt - \cos 3nt\}$$

$$\frac{d^2 \cdot \sin^4 nt}{(ndt)^2} = 2\cos 2nt - 2\cos 4nt, \ \&\text{c.}$$

thus[24]

$$\begin{aligned}\frac{r}{a} &= 1+\frac{e^2}{2} - e\cos nt - \frac{e^2}{2}\cos 2nt \\ &- \frac{e^2}{1.2.2^2}\cdot\{3\cos 3nt - 3\cos nt\} \\ &- \frac{e^4}{1.2.3.2^3}\cdot\{4^2\cos 4nt - 4.2^2\cos 2nt\} \\ &- \frac{e^5}{1.2.3.4.2^4}\cdot\left\{5^3\cos 5nt - 5.3^3\cos 3nt + \frac{5.4}{1.2}\cos nt\right\} \\ &- \text{\&c. \&c. \&c.}\end{aligned}$$

This gives a value of the radius vector in functions of the time.

Kepler's Problem. To find a Value of the true Anomaly in functions of the Mean Anomaly

388. The determination of v in terms of nt is Kepler's problem of finding the true anomaly in terms of the mean anomaly; or, to divide the area of a semicircle in a given ratio by a line drawn from a given point in the diameter–in order to accomplish this, a value of v in functions of u must be obtained from

$$\tan\tfrac{1}{2}v = \sqrt{\frac{1+e}{1-e}}\cdot\tan\tfrac{1}{2}u\,;$$

therefore let

$$l = \frac{e}{1+\sqrt{1-e^2}}\,,$$

then

$$e = \frac{2l}{1+l^2},\ \text{and}\ \sqrt{\frac{1+e}{1-e}} = \frac{1+l}{1-l}\,.$$

Again,

$$\sin\tfrac{1}{2}v = c^{v\sqrt{-1}}-1,\ \cos\tfrac{1}{2}c^{v\sqrt{-1}}+1,$$

c being the number whose logarithm is unity; hence the equation in question becomes

$$\frac{c^{v\sqrt{-1}}-1}{c^{v\sqrt{-1}}+1} = \frac{1+l}{1-l}\cdot\frac{c^{u\sqrt{-1}}-1}{c^{u\sqrt{-1}}+1}\,,$$

whence

$$c^{v\sqrt{-1}} = \frac{1-l\,c^{-u\sqrt{-1}}}{1-l\,c^{u\sqrt{-1}}}\cdot c^{u\sqrt{-1}}\,;$$

or taking its logarithm,

$$v = u + \frac{\log\left\{1 - l\,c^{-u\sqrt{-1}}\right\} - \log\left\{1 - l\,c^{u\sqrt{-1}}\right\}}{\sqrt{-1}}.$$

Or

$$v = u + l\left\{\frac{c^{u\sqrt{-1}} - c^{-u\sqrt{-1}}}{\sqrt{-1}}\right\} + \frac{l^2}{2}\left\{\frac{c^{2u\sqrt{-1}} - c^{-2u\sqrt{-1}}}{\sqrt{-1}}\right\} + \&\text{c.}$$

but

$$\frac{c^{mu\sqrt{-1}} - c^{-mu\sqrt{-1}}}{2\sqrt{-1}} = \sin mu\,;$$

m being any whole positive number, therefore

$$v = u + 2l\sin u + \frac{2l^2}{2}\sin 2u + \frac{2l^3}{3}\sin 3u + \&\text{c.}$$

The true anomaly may now be found in terms of the mean anomaly.

389. In order to have *v* in terms of the mean anomaly and of the powers of *e*, values of *u*, $\sin u$, $\sin 2u$, must be found in terms of the sines of *nt* and its multiples; and l, l^2, &c. must be developed into series according to the powers of *e*. Both may be accomplished by *Lagrange's*[25] Theorem,[26] for if

$$f = \frac{1}{a} = \frac{1}{1+\sqrt{1-e^2}} = \frac{l}{e};$$

when

$$e = 0,\ a = 2,\ f' = \frac{1}{2},\ \frac{df'}{da} = -\frac{1}{2^2}$$

so that

$$f = \frac{l}{e} = \frac{1}{2}\left\{1 + \left(\frac{e}{2}\right)^2 + \frac{4}{2}\left(\frac{e}{2}\right)^4 + \frac{5\,.\,6}{2\,.\,3}\left(\frac{e}{2}\right)^6 + \&\text{c.}\right\}$$

or generally

$$f' = \frac{1}{2^i},\ \frac{df'}{da} = -\frac{1}{2^{i+1}},$$

consequently

$$l^i = \frac{e^i}{2^i}\left\{1 + i\left(\frac{e}{2}\right)^2 + \frac{i(i+3)}{2}\left(\frac{e}{2}\right)^4 + \frac{i(i+3)(i+5)}{2\cdot 3}\left(\frac{e}{2}\right)^6 + \&\text{c.}\right\}$$

If *i* be successively assumed to be 1, 2, 3, &c., this equation will give all the powers of l in series, ascending according to the powers of *e*.

Again. If we assume $f = u = nt + e\sin u$, f is a function of u which is a function of *e*; hence

$$\frac{df}{de} = \frac{df}{du} \cdot \frac{du}{de};$$

and as $f' = nt$, when $e = 0$, so $\frac{du}{de} = \sin nt$. And[27] $\frac{df'}{de} = \sin nt \cdot \frac{df}{du}$. Whence by the same process it will be found that[28]

$$u = f + e \sin nt \cdot \frac{df}{ndt} + \frac{e^2}{2} \cdot \frac{d \cdot \sin^2 nt \cdot df}{(ndt)^2} + \frac{e^3}{2 \cdot 3} \cdot \frac{d^2 \cdot \sin^2 nt \cdot df}{(ndt)^3} + \text{\&c. \&c.}$$

Values of u, $\sin u$, $\sin 2u$, &c., may be determined from this expression by making f successively equal to nt, $e \cdot \sin nt$, &c. The substitution of these, and of the powers of l, will complete the development of v, but the same may be effected very easily from the expression $dv = \frac{hdt}{r^2}$ of article 372, or rather from

$$dv = \sqrt{1 - e^2} \cdot \frac{a^2}{r^2} \cdot ndt.$$

390. If $r^i = a^i (1 - e \cos nt)^i$ be put for $r = a(1 - e \cos nt)$, and $ia^i (1 - e \cos nt)^{i-1} \cdot e \sin nt$ for $\frac{dr}{ndt}$ in the development of r in article 387, it becomes[29]

$$\begin{aligned} \frac{r^i}{a^i} &= (1 - e \cos nt)^i + i \cdot e^2 \cdot \sin^2 . nt (1 - e \cos nt)^{i-1} \\ &+ \frac{i \cdot e^3 \cdot \sin^3 . nt (1 - e \cos . nt)^{i-1}}{2ndt} \\ &+ \frac{i \cdot e^4 d^2 \cdot \sin^4 . nt (1 - e \cos . nt)^{i-1}}{2 \cdot 3 \cdot n^2 dt^2} + \text{\&c.} \end{aligned}$$

whatever i may be. Let $i = -2$, then

$$\begin{aligned} \frac{a^2}{r^2} &= 1 + 2e . \cos . nt + \frac{e^2}{1 \cdot 2} \cdot (1 + 5 . \cos . nt) \\ &+ \frac{e^3}{1 \cdot 2^2} (13 . \cos . 3nt + 3 . \cos . nt) \\ &+ \frac{e^4}{1 \cdot 2^3 \cdot 3} (103 . \cos . 4nt + 8 . \cos . 2nt + 9) + \text{\&c.} \end{aligned}$$

If this quantity be substituted in the preceding expression for dv, when the integration is accomplished, and the approximation only carried to the sixth powers of e, the result will be

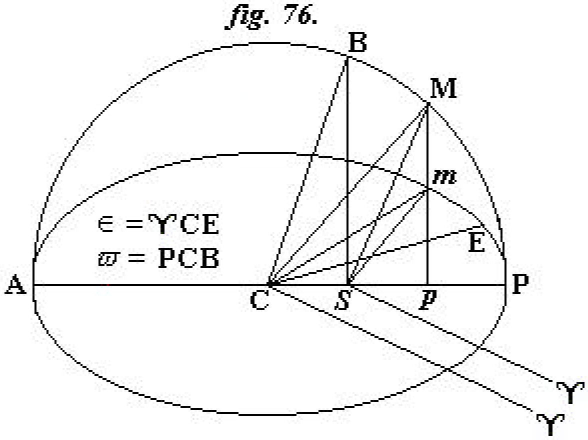
$$
\begin{aligned}
v = nt &+ \left\{2e - \frac{1}{4}\cdot e^3 + \frac{5}{96}\cdot e^5\right\}\sin nt \\
&+ \left\{\frac{5}{4}\cdot e^2 - \frac{11}{24}\cdot e^4 + \frac{17}{192}\cdot e^6\right\}\sin 2nt \\
&+ \left\{\frac{13}{12}\cdot e^3 - \frac{45}{64}\cdot e^5\right\}\sin 3nt \\
&+ \left\{\frac{103}{96}\cdot e^4 - \frac{451}{480}\cdot e^6\right\}\sin 4nt\,, + \&\text{c. } \&\text{c.}
\end{aligned}
$$

391. The angles v and nt which are the true and mean anomaly, begin at the perihelion; but if they be estimated from the aphelion, it will only be necessary to make e negative in the values of r and v, or to add 180° to each angle. This expression gives $v - nt$ the equation of the centre.

True Longitude and Radius Vector in functions of the Mean Longitude

392. Instead of fixing the origin of the time at the instant of the planet's passage at the perihelion, let it be fixed at any point whatever, as E, fig. 76,[30] so that $nt = \text{ECB}$, then by adding the constant angle ♈CE represented by $\in$, the whole angle ♈CB $= nt + \in$ is the mean longitude of the planet, ♈ being the equinox of Spring; and if the constant angle ♈CP, which is the longitude of the perihelion, be represented by v, the angle $nt + \in - v = \text{PCB}$ must be put for nt, and if v be estimated from ♈, then $v - v$ must be put for v, and the preceding values of v and r become,

fig. 76.



$$
\begin{aligned}
v = nt + \in &+ \left\{2e - \frac{1}{4}e^3\right\}\sin\left(nt + \in - v\right) \\
&+ \left\{\frac{5}{4}e^2 - \frac{11}{24}e^4\right\}\sin 2\left(nt + \in - v\right) + \&\text{c. } \&\text{c.}
\end{aligned} \tag{97}
$$

$$
\begin{aligned}
\frac{r}{a} = 1 + \frac{1}{2}e^2 &- \left\{e - \frac{3}{8}e^3\right\}\cos\left(nt + \in - v\right) \\
&- \left\{\frac{1}{2}e^2 - \frac{1}{3}e^4\right\}\cos 2\left(nt + \in - v\right) - \&\text{c. } \&\text{c.}
\end{aligned} \tag{98}
$$

393. v is the true longitude of the planet and $nt+\in$ its mean longitude both being estimated on the plane of the orbit. The angle $\in = ♈$CE is the longitude of the point E, from whence the time is estimated, commonly called the longitude of the epoch.

394. In astronomical series, the quantities which multiply the sines and cosines are the coefficients; and the angles are called the arguments: for example in[31]

$$\left\{2e-\frac{1}{4}e^{3}\right\}\sin\left(nt+\in-v\right)$$

the part $\left\{2e-\frac{1}{4}e^{3}\right\}$ is the coefficient, and $\left(nt+\in-v\right)$ is the argument.

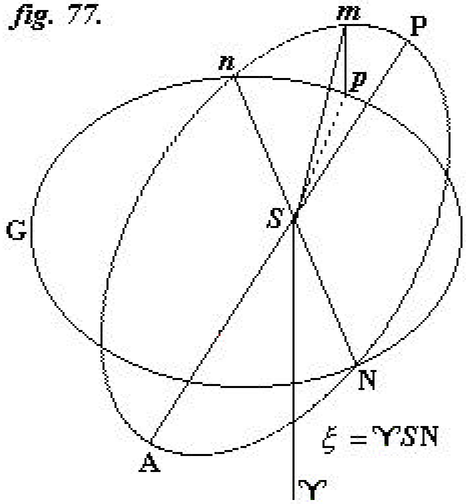
395. Although the time increases without limit, these series converge: for, as a sine or cosine never can exceed the radius, the values of the sines and cosines in these series never can be greater than unity, however much the time may increase, and as the powers of e soon become extremely small, they converge rapidly.

396. The values of v and r answer for all the planets and satellites, since they are independent of the masses, for the mass of a planet is so inconsiderable in comparison of that of the sun, that it may be omitted, and as the mass of the sun forms the standard of comparison for the masses of the other bodies of the system, it is assumed to be the unit of measure. The same holds with regard to a planet and its satellites.

Determination of the Position of the Orbit in space

397. The values of v and r give the place of a body in its orbit, but not its position in space; they however afford the means of ascertaining it. For let N*pn*G, fig. 77,[32] be the plane of the ecliptic, or fixed plane at the epoch, on which the plane of the orbit P*n*AN has a very small inclination; then N*n* is the line of the nodes; S the sun, and if mp be a perpendicular from the planet on the plane of the ecliptic, it will be the tangent of the latitude mSp. Let ♈*S*N the longitude of the node be represented by x when estimated on the plane of the orbit, and let q represent the same angle when projected on the plane of the ecliptic; also let $v_{/} = ♈Sp$ be the true longitude ♈*Sm* or v, when projected on the plane of the ecliptic. Then

fig. 77.



$$\text{N}Sp = v_{/} - q\,,\quad \text{N}Sm = v - z_{/}\,.$$

And if f be the inclination of the two planes, it appears from the right angled triangle pNm, that

$$\tan\left(v_{/}-q\right)=\cos f\tan\left(v-x\right). \tag{99}$$

Projected Longitude in Functions of true Longitude

398. This gives $v_{/}$ in terms of v, and the contrary. But these two angles may be obtained in terms of one another in very converging series by means of the expression,

$$\tfrac{1}{2}v=\tfrac{1}{2}u+l\sin u+\frac{l^2}{2}\sin 2u+\frac{l^3}{3}\sin 3u+\&c.$$

which was derived from $\tan\tfrac{1}{2}v=\sqrt{\frac{1+e}{1-e}}\cdot\tan\tfrac{1}{2}u$, by making $l=\frac{e}{1+\sqrt{1-e^2}}$. If $v_{/}-q$ be put for $\tfrac{1}{2}v$, $v-z$ for $\tfrac{1}{2}u$, and $\cos f$ for $\sqrt{\frac{1+e}{1-e}}$; then

$$l=\frac{\cos f-1}{\cos f+1}=-\tan^2\tfrac{1}{2}f,$$

and the series becomes

$$v_{/}-q=v-x-\tan^2\tfrac{1}{2}f\cdot\sin 2\left(v-x\right)+\tfrac{1}{2}\tan^4\tfrac{1}{2}f\cdot\sin 4\left(v-x\right)-\&\text{c}. \tag{100}$$

True Longitude in Functions of projected Longitude

On the contrary, if $v-x$ be put for $\tfrac{1}{2}v$, and $v_{/}-q$ for $\tfrac{1}{2}u$, the result will be

$$v-x=v_{/}-q+\tan^2\tfrac{1}{2}f\cdot\sin 2\left(v_{/}-q\right)+\tfrac{1}{2}\tan^4\tfrac{1}{2}f\cdot\sin 4\left(v_{/}-x\right)+\&\text{c}. \tag{101}$$

Projected Longitude in Functions of Mean Longitude

399. A value of $v_{/}-q$, or N*Sp*, may be found in terms of the sines and cosines of *nt*, and its multiple arcs, from the series

$$v=nt+\in+\{2e-\tfrac{1}{4}e^3\}\sin\left(nt+\in-v\right)+\left\{\frac{5}{4}e^2-\frac{11}{24}e^4\right\}\sin 2\left(nt+\in-v\right)+\&\text{c}.$$

which may be written

$$v=nt+\in+e\text{Q}.$$

If x be subtracted from both sides of this equation, and the sines taken in place of the arcs, it becomes

$$\sin(v-x)=\sin(nt+\in-x+e\mathrm{Q}),$$

which may be expanded into a series, ascending, according to the powers of e, by the method already employed for the development of v and r; if

$$f=\sin(v-x)=\sin(nt+\in-x+e\mathrm{Q}).$$

Whence it may be found that,

$$\sin i(v-x)=\sin i(nt+\in-x+e\mathrm{Q})=\left\{1-\frac{i^2e^2\mathrm{Q}^2}{1.2}+\frac{i^4e^4\mathrm{Q}^4}{1.2.3.4}-\&\mathrm{c.}\right\}\times\sin i(nt+\in-x)$$
$$+\left\{ie\mathrm{Q}-\frac{i^3e^3\mathrm{Q}^3}{1.2.3}+\frac{i^5e^5\mathrm{Q}^5}{1.2.3.4.5}-\&\mathrm{c.}\right\}\times\cos i(nt+\in-x)+\&\mathrm{c.}$$

Latitude

400. If *mp*, the tangent of the latitude, be represented by s, the right-angled triangle *m*N*p* gives

$$s=\tan f\sin(v_{/}-q).$$

Curtate Distances

401. Let *r*, be the curtate distance *Sp*, then *Spm*, being a right angle,

$$Sp:Sm::1:\sqrt{1+s^2};$$

hence

$$Sp=\frac{Sm}{\sqrt{1-s^2}};$$

or

$$r_{/}=r(1+s^2)^{-\frac{1}{2}}=r\left\{1-\tfrac{1}{2}s^2+\tfrac{3}{8}s^4-\&\mathrm{c.}\right\} \qquad \textbf{(102)}$$

402. Thus $v_{/}$, s, and $r_{/}$, the longitude, latitude, and curtate distance[33] of the planet are determined in convergent series of the sines and cosines of *nt* and its multiples; if therefore the time be assumed, the place of the body will be known, and the means are thus furnished for computing tables of the motions of the planets and satellites, from which their elliptical places may be ascertained at any instant.

403. A particular period is chosen as an origin from whence the time is estimated, which is called the Epoch of the tables: the elements of the orbits are determined by observation; and the longitude, latitude, and distance of the body from the sun are computed for that period, and for every succeeding day, hour, and minute, if necessary, for any number of years; these are arranged in tables according to the time; so that by inspection alone the corresponding place of the body referred to the fixed plane, or position of the ecliptic at the epoch, may be found.

Fortunately for the facility of astronomical calculations, the orbits of the celestial bodies are either very nearly circular, as in the planets and satellites, or very eccentric, as in the comets. In both circumstances the series which determines the motion of the body may be made to converge rapidly, which would not be the case if the eccentricity bore a mean ratio to the greater axis.

Motion of Comets

404. If the ratio of the eccentricity to the greater axis be made very nearly equal to unity, instead of a very small fraction, the preceding series will then give the place of a comet in a very eccentric orbit, with this difference, that the terms have the increasing powers of the difference between unity and the ratio of the eccentricity to the greater axis, as coefficients, instead of the powers of that ratio itself. This difference is zero in the parabola; then the value of the radius vector becomes

$$r = \frac{\mathrm{D}}{\cos^2 . \frac{1}{2} v},$$

D being the perihelion distance: hence, in the parabola, the distance *Sm* is equal to the perihelion distance *S*P, divided by the square of the cosine of half the true anomaly P*Sm*. If, then, the true anomaly were known, the distance of the comet from the sun would be determined from this equation. When the body moves in a parabola, the equation between the mean and true anomaly is reduced to a cubic equation between the time and the tangent of half the true anomaly *PSm*.

Arbitrary Constant Quantities of Elliptical Motion, or Elements of the Orbits

405. There are six elements in the orbit of each celestial body: four of elliptical motion, namely, the mean distance of the planet from the sun; the eccentricity; the mean longitude of the planet at the epoch; and the longitude of the perihelion at the same epoch. The other two elements relate to the position of the orbit in space, namely, the longitude of the ascending mode at the epoch, and the inclination of the orbit on the plane of the ecliptic. The mean values of all these must be determined by observation, before the motion of the body call be ascertained, or tables computed. Hence there are forty-two elements to be determined for the seven principal planets, and twenty-four more for the four new planets, Ceres, Pallas, Juno, and Vesta,[34] besides those of the moon and satellites. Tables have been computed for most of these bodies; some of the satellites, however, are but little known, and the theory of the four new planets is still imperfect.

The same series that determine the motions of the planets answer equally well for the elliptical motion of the moon and satellites, only the mass of the planet is to be employed in place of that of the sun, omitting the mass of the satellite.

Co-ordinates of a Planet

406. The simplicity of analytical expressions very much depends on a skilful choice of co-ordinates, which are arbitrary and infinite in number, but so connected, that any one set may be expressed in values of any other. For example, the place of the planet *m* has been determined by the angles ♈*Sm*, *mSp*, and *Sm*, fig. 77, but these have been changed into ♈*Sp*, *pSm*, and *Sp*, which are the heliocentric longitude, latitude, and curtate distance of *m*. Again, from the latter, the geocentric longitude, latitude, and distance may be deduced, that is, the place of *m* as seen from the earth; and, lastly, the right ascension and declination of *m*, or its place referred to the equator, may be obtained from its geocentric longitude and latitude.

These quantities are given in terms of the mean longitude or time, since the first co-ordinates are given in series of the sines and cosines of that quantity. In the theory of the moon, the series are found to converge more rapidly, if the mean longitude, latitude, and distance are determined in functions of the true longitude. All these co-ordinates are connected by spherical triangles, so they are easily deduced from one another.

Determination of the Elements of Elliptical Motion

407. Were the primitive velocity with which the bodies of the solar system projected in space known, the values of the elements of their orbits might be determined; for if the equation (90) be resumed, and if the first member, which is the square of the velocity, be represented by V^2, then

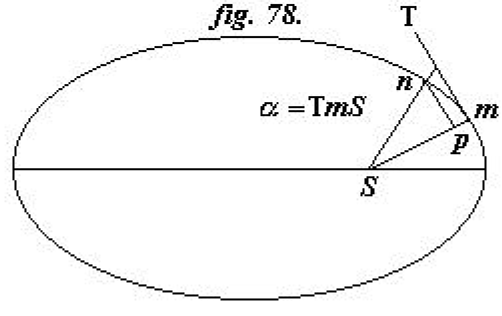
$$V^2 = m\left\{\frac{2}{r} - \frac{1}{a}\right\}$$

in which r is the radius vector, and a is half the greater axis of the conic section, m being the masses of the sun and planet. Thus the velocity is independent of the eccentricity of the orbit.

If u be the angular velocity which the planet would have if it described a circle at the distance of unity round the sun, then $r = a = 1$, and the preceding expression gives $u^2 = m$; hence

$$V^2 = u^2\left\{\frac{2}{r} - \frac{1}{a}\right\},$$

fig. 78.



V being the primitive velocity with which the body moved in a conic section. This equation will give a value of a by means of the primitive velocity of m, and its distance from S, fig. 78.[35] a is positive in the ellipse, infinite in the parabola, and negative in the hyperbola;

thus the orbit of m is an ellipse, a parabola, or hyperbola, according as V is less, equal to, or greater than $u=\sqrt{\frac{2}{r}}$. It is remarkable that the *direction* of the primitive impulse has no influence on the nature of the conic section in which the planet moves; the intensity alone has that effect.

To determine the eccentricity of the orbit, let a be the angle T*mS*, that the direction of the relative motion of *m* makes with the radius vector *r*; then

$$mn : mv :: ds : dr :: 1 : \cos a\ ;$$

then

$$\frac{ds}{dt}\cos a=\frac{dr}{dt},\ \text{but}\ \frac{ds}{dt}=V\ ,$$

hence

$$V^2\cos^2 a=\frac{dr^2}{dt^2};\ \text{or if}\ m\left\{\frac{2}{r}-\frac{1}{a}\right\}$$

be put for V,

$$\frac{dr^2}{dt^2}=m\left\{\frac{2}{r}-\frac{1}{a}\right\}\cos^2 a\ ;$$

but by article 377,

$$2mr-\frac{mr^2}{a}-\frac{r^2dr^2}{dt^2}=ma\left(1-e^2\right);$$

hence

$$a\left(1-e^2\right)^2=r^2\sin^2 a\left\{\frac{2}{r}-\frac{1}{a}\right\},$$

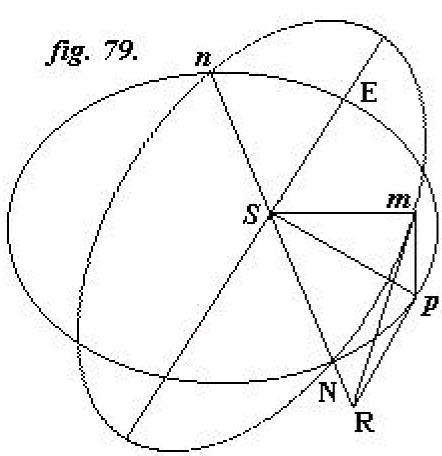
which gives the eccentricity of the orbit. The equation of conic sections,

$$r=\frac{a\left(1-e^2\right)}{1+e\cos v}$$

gives

$$\cos v=\frac{a\left(1-e^2\right)-r}{er}.$$

fig. 79.



Thus the angle v, that the radius vector makes with the perihelion distance, is found, and, consequently, the position of the perihelion. The equations (96) will then give the angle *u*, or eccentric anomaly, and, by means of it, the instant of the passage at the perihelion.

In order to have the position of the orbit, with regard to a fixed plane passing through the centre of *S*, fig. 77, supposed immovable,[36] let f be the inclination of the two planes, and $x = m$*S*N; also let $mp = z$ be the primitive elevation of the planet above the fixed plane, which is[37] supposed to be known;

then

$$r \sin x \sin f = z\,.$$

So that f, the inclination of the orbit, will be known when x shall be determined. For that purpose, let $l = m\mathrm{R}p$, fig. 79, be the angle made by mR, the primitive direction of the relative motion of m with the plane ENB; then the triangle $mS\mathrm{R}$, in which $Sm\mathrm{R} = a$, $NSm = x$, and $Sm = r$, gives

$$m\mathrm{R} = \frac{r \sin x}{\sin(x + a)};$$

then

$$\frac{z}{m\mathrm{R}} = \sin l\,,$$

which is given, because l is supposed to be known; therefore

$$\tan x = \frac{z \sin a}{r \sin l - z \cos a}\,.$$

The elements of the orbit of the planet being determined by these formulae in terms of r, z, the velocity of the planet, and the direction of its motion, the variations of these elements, corresponding to the supposed variations in the velocity and its direction, may be obtained; and it will be easy, by means of methods that will be hereafter given, to have the differential variations of these elements, arising from the action of the disturbing forces.

Velocity of Bodies moving in Conic Sections

408. As the actual motions of the bodies of the solar system afford no information with regard to their primitive motions, the elements of their orbits can only be known by observation; but when these are determined, the velocities with which the bodies of the solar system were first projected in space, may be ascertained. If the equation

$$V^2 = u^2\left\{\frac{2}{r} - \frac{1}{a}\right\}$$

be resumed, then in the circle $r = a$, since the eccentricity is zero; hence[38]

$$V = u\sqrt{\frac{1}{r}};$$

therefore

$$V : u :: 1 : \sqrt{r}\,.$$

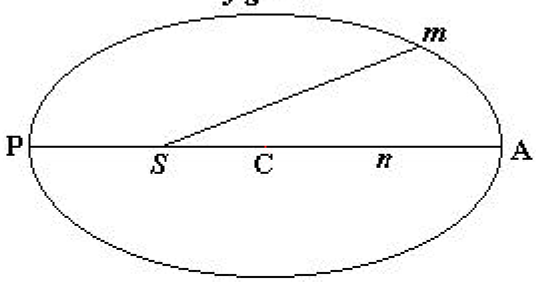
Thus the velocities of planets in different circles are as the square roots of their radii.

In the parabola, a is infinite; hence

$$\frac{1}{a} \text{ is zero, and } V = \sqrt{\frac{2}{r}}.$$

Thus the velocities in different points of a parabolic orbit are reciprocally as the square roots of the radii vectores, and the velocity in each point is to the velocity the planet would have if it moved in a circle with a radius equal to r, as $\sqrt{2}$ to 1.

fig. 80.



409. When an ellipse is infinitely flattened, it becomes a straight line; hence, in this case, V will express the velocity of m, if it were to descend in a straight line towards the sun; for then Sm, fig. 80, would coincide with SA. If m were to begin to fall from a state of rest at A, its velocity would be zero at that point; hence $\frac{2}{r} - \frac{1}{a} = 0$. Now, suppose that, in falling from A to n, the body had acquired the velocity V, then the equation would be

$$V^2 = u^2 \left\{ \frac{2}{r'} - \frac{1}{a} \right\},$$

and eliminating a, which is common to the two last equations,

$$V = u\sqrt{\frac{2(r - r')}{rr'}},$$

in which $r' = Sn$. This is the relative velocity the body m has acquired in falling from A through $r - r' = \mathrm{A}n$. Imagine the body m to have acquired, by its fall through An, the same velocity with a body moving in a conic section; the velocity of the latter body is

$$V' = u\sqrt{\frac{2}{r} - \frac{1}{u}}.$$

If these two be equated,

$$\mathrm{A}n = (r - r') = \frac{r(2a - r)}{4a - r}.$$

This expression gives the height through which a body moving in a conic section must fall, from the extremity A of the radius vector, in order to acquire the relative velocity which it had at A.

In the circle $a = r$, hence $\mathrm{A}n = \frac{1}{3}r$; in the ellipse, A$n$ is less than $\frac{1}{2}r$;[39] in the parabola, a is infinite, which gives $\mathrm{A}n = \frac{1}{2}r$; and in the hyperbola a is negative, and therefore An is greater than $\frac{1}{2}r$.

Notes

[1] *curtate*. Shortened or reduced;—said of the distance of a planet from the sun or earth, as measured in the plane of the ecliptic, or the distance from the sun or earth to that point where a perpendicular, let fall from the planet upon the plane of the ecliptic, meets the ecliptic. *Webster's Revised Unabridged, 1913.*

[2] This *reads* "article 146" in the 1st edition (published erratum).

[3] Pontécoulant, Philippe Gustave le Doulcet, 1795-1874, French astronomer and graduate of the École Polytechnique.

[4] The second result below *reads* $md\frac{x}{r}$ in the 1st edition.

[5] This *reads* "article 269" in the 1st edition.

[6] The left hand side uses the letter *o* rather than the numeral 0 in the 1st edition.

[7] *horary*. Of or pertaining to an hour; noting the hours. *Webster's Revised Unabridged, 1913 Edition.*

[8] The 1st edition uses the symbol g here rather than symbol ♈ introduced in the article 359.

[9] This *reads* "269" in the 1st edition (published erratum).

[10] The denominator *reads* $1 - e\cos(v - v\ )$ in the 1st edition (published erratum).

[11] This *reads* $l = 0$ in the 1st edition (published erratum).

[12] Note that the parameter n is used here for the first time.

[13] Or $\mathrm{T} = \frac{360^\circ}{n}$, since $n = \frac{\sqrt{u}}{a^{\frac{3}{2}}}$ as defined in article 383.

[14] Colin Maclaurin (1698-1746). Maclaurin's *theorem* is a special case of Taylor's series (see note 18, *Book I, Chapter VII*) now named after him. The Taylor series for the function f at $\mathrm{x} = \mathrm{c}$ is $\sum_{n=0}^{\infty} \frac{\mathrm{f}^{(n)}(\mathrm{c})}{n!}(\mathrm{x} - \mathrm{c})^{n}$. The Taylor series for the function f at $\mathrm{x} = 0$ is also called a Maclaurin series for f.

[15] This *reads* $\frac{d^2u'}{de^2} = \cos nt \sin nt$ in the 1st edition (published erratum).

[16] A semicolon is omitted from second line below and the multiplier symbol added to several terms for consistency.

[17] $\frac{r}{a}$ is used for r in the 1st edition (published erratum).

[18] $\frac{dr'}{de}$ is used for $\frac{dr}{de}$ in the 1st edition (published erratum). However, the original equation reads $\frac{dr'}{de} = \frac{dr}{de} + \frac{dr}{du}\cdot\frac{du}{de}$; consequently, the corrected expression should read $\frac{dr}{de} = \frac{dr'}{de} + \frac{dr'}{du}\cdot\frac{du}{de}$.

[19] Punctuation added.

[20] $\cos nt$ is used for $a\cos nt$ in the next three equations in the 1st edition (published erratum).

[21] This *reads* "become" in the 1st edition.

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[22] Right hand side *reads* $\dfrac{d^2\left(\sin^3 nt \cdot \dfrac{dr}{ndt}\right)}{(ndt)^2}$ in the 1st edition.
[23] Equations modified to include multiplier symbol in relevant terms.
[24] Punctuation removed after first, second, third and fourth terms.
[25] see note 16, *Preliminary Dissertation*.
[26] *Lagrange's Theorem:* The theorem on the development of a function in series, for examples if y=x+ef(x), where e is small, then the theorem of Legrange gives the development of f(y) in series of y.
[27] This *reads* $\dfrac{df'}{de} = \sin nt \dfrac{df}{du}$ in the 1st edition.
[28] Equation modified to include multipliers.
[29] The remaining three equations in article 390 are modified slightly from the 1st edition text to reflect a consistent inclusion of the multiplier symbol where appropriate.
[30] Fig. 76 is modified from the 1st edition with inclusion of explicit definitions of $\in$ and v .
[31] This is the third term in equation (97). The presentation here is altered from the 1st edition with inclusion of the parentheses for clarity.
[32] Fig. 77 is modified from the 1st edition to include an explicit definition of x .
[33] See note 1.
[34] See note 9, *Preliminary Dissertation*.
[35] Figure 78 is modified to include an explicit definition of a .
[36] Spelled "immoveable" in the 1st edition.
[37] Typographical error in the 1st edition omits the letter "i" in "is".
[38] The symbol *V* reads *v* in the 1st edition (published erratum).
[39] Punctuation changed from a colon in the 1st edition to a semicolon.