BOOK II

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CHAPTER VIII

PERTURBATIONS OF THE PLANETS IN LONGITUDE, LATITUDE, AND DISTANCE

532. THE position of a planet in space is fixed when its curtate distance *Sp*, fig. 77, its projected longitude γSp, and its latitude *pm*, are known. The determination of these three co-ordinates in functions of the time is the principal object of Physical Astronomy; these quantities in series ascending according to the powers of the eccentricities and inclinations are given in article 399, and those following, supposing the planet to move in a perfect ellipse; but if values of the elements of the orbits corrected by their periodic and secular variations be substituted instead of their elliptical elements, the same series will determine the motion of the planet in its real perturbed orbit.

533. The projected longitude and curtate distance only differ from the true longitude and distance on the orbit by quantities of the second order with regard to the inclinations; and when the orbit at the epoch is assumed to be the fixed plane, these quantities as well as those of the latitude that depend on the product of the inclination by the eccentricity are so small that they are insensible, as will readily appear if it be considered that any inclination the orbit may have acquired subsequently to the epoch, can only have arisen from the small secular variation in the elements; besides the epoch may be chosen to make it so, being arbitrary. Hence the perturbations in the longitude and radius vector may be determined as if the orbits were in the same plane, and the latitude may be found in the hypothesis of the orbits being circular, provided the orbit at the epoch be taken as the fixed plane: circumstances which greatly facilitate the determination of the perturbations.

The following very elegant method of finding the perturbations, by considering the troubled orbit as an ellipse whose elements are varying every instant, was employed by Lagrange;[1] but Laplace's method,[2] which will be explained afterwards, has the advantage of greater simplicity, especially in the higher approximations.

534. In the elliptical hypothesis the radius vector and true longitude are expressed, in article 392, by

$$r = \text{functions}.\left(r, z, e, \in, v\right),$$
$$v = \text{functions}.\left(z, e, \in, v\right),$$

but in the true orbit these quantities become

$$a + da,\ z + dz,\ e + de,\ v + dv;$$

therefore

$$d\,r = \frac{dr}{da}\cdot da + \frac{dr}{dz}\cdot dz + \frac{dr}{de}\cdot d\,e + \frac{dr}{d\in}\cdot d\in + \frac{dr}{d\,v}\cdot d\,v\,,$$
$$d\,v = \frac{dv}{dz}\cdot dz + \frac{dv}{de}\cdot d\,e + \frac{dv}{d\in}\cdot d\in + \frac{dv}{d\,v}\cdot d\,v\,;$$

and if the values of the periodic variations in the elements in article 529 be substituted instead of $d\,a$, $d\,z$, &c., the perturbations in the radius vector and true longitude will be obtained; the approximation extending to the first powers of the eccentricities and inclinations inclusively.

535. The perturbations in longitude way be expressed under a more simple form; for by article 372,

$$dv = \frac{\sqrt{a\left(1-e^2\right)}}{r^2}\cdot dt,$$

an equation belonging both to the elliptical and to the real orbit, since it is a differential of the first order; on that account it ought not to change its form when the elements vary; hence

$$d\,.\,d\,v = \tfrac{1}{2}\sqrt{\frac{1-e^2}{a}}\cdot\frac{d\,a}{r^2}\cdot dt - \sqrt{\frac{a}{1-e^2}}\cdot\frac{e d\,e}{r^2}\cdot dt - 2\sqrt{a\left(1-e^2\right)}\,\frac{d\,r}{r^3}\cdot dt;$$

and neglecting the squares of the disturbing forces, the integral is

$$d\,v = \frac{1}{2a}\cdot\int d\,a\,.dv - \frac{e}{1-e^2}\cdot\int de\,.\,dv - 2\int\frac{d\,r}{r}\cdot dv\,.$$

But

$$h = \sqrt{a\,.\left(1-e^2\right)},$$

then

$$\frac{d\,h}{h} = \frac{1}{2a}\cdot d\,a - \frac{e}{1-e^2}d\,e\,;$$

therefore

$$d\,v = \int\left(\frac{d\,h}{h} - \frac{2d\,r}{r}\right)\cdot dv \qquad \textbf{(148)}$$

will give the perturbations in longitude when those in the radius vector are known.

Perturbations in the Radius Vector

536. By article 392,

$$r = a\left(1+\tfrac{1}{2}e^2 - e\cos\left(nt+\in -v\right)-\tfrac{1}{2}e^2\cos 2\left(nt+\in -v\right)\right);$$

whence

$$dr = da - a\,de\cos\left(nt+\in -v\right) - aed v\,\sin\left(nt+\in -v\right)\left(1+2e\cos\left(nt+\in -v\right)\right)$$
$$-3ed\,a\cos\left(nt+\in -v\right)+2aed\,e+ae\left(dz+d\in\right)\sin\left(nt+\in -v\right).$$

If the values of da, de, dv, dz, $d\in$, from article 529, be substituted in this expression, after the reduction of the products of the sines and cosines to the cosines of multiple arcs, and substitution for M_0, M_1, N_3, N_4, N_5, from article 459, it becomes

$$\begin{aligned}\frac{dr}{a} &= \frac{m'}{2}\cdot\Sigma.C_i.\cos i\left(n't-nt+\in'-\in\right)\\ &+m'.e.\Sigma.D_i.\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in -v\right\}\\ &+m'.e'.\Sigma.E_i.\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in -v'\right\},\end{aligned} \qquad \textbf{(149)}$$

where

$$C_i = \frac{n^2}{n^2-i^2\left(n'-n\right)^2}\left\{\frac{2n}{n-n'}aA_i + a^2\frac{dA_i}{da}\right\},$$

$$D_i = \frac{n^2}{\left\{i\left(n'-n\right)+n\right\}^2-n^2}\left\{\frac{3n}{n'-n}aA_i - \frac{i^2\left(n'-n\right)\left\{i\left(n'-n\right)-n\right\}-3n^2}{n^2}C_i+\tfrac{1}{2}a^3\left(\frac{d^2A_i}{da^2}\right)\right\},$$

$$E_i = \frac{n^2}{\left\{i\left(n'-n\right)+n\right\}^2-n^2}\left\{\frac{(i-1)(2i-1)n}{i\left(n'-n\right)+n}\cdot aA_{(i-1)} - \frac{i^2\left(n'-n\right)+n}{i\left(n'-n\right)+n}a^2\frac{dA_{(i-1)}}{da}-\tfrac{1}{2}a^3\frac{d^2A_{(i-1)}}{da^2}\right\}.$$

The Perturbations in Longitude

537. Having thus determined the perturbations in the radius vector, the term $\frac{2dr}{r}$ is known; and if substitution be made for da and de, from article 529, $\frac{dh}{h}$ will be obtained, and the integral of equation (148) will give[3]

$$\begin{aligned}dv &= \frac{m'}{2}\Sigma.F_i.\sin i\left(n't-nt+\in'-\in\right)\\ &+m'e.\Sigma.G_i.\sin\left\{i\left(n't-nt+\in'-\in\right)+nt+\in -v\right\}\\ &+m'e'.\Sigma\,H_i.\sin\left\{i\left(n't-nt+\in'-\in\right)+nt+\in -v'\right\}.\end{aligned}$$

Where

$$F_i = \frac{n}{i(n-n')}\left\{-\frac{n}{n-n'}\cdot aA_i + 2C_i\right\}$$

$$G_i = \frac{n}{i(n'-n)+n}\left\{\frac{(i-1)n}{n'-n}\cdot aA_i - 2D_i - \frac{i\{i(n'-n)-n\}+6n}{2n}\cdot C_i\right\}$$

$$H_i = \frac{n}{i(n'-n)+n}\times\left\{\frac{-(i-1)(2i-1)naA_{(i-1)} - (i-1)na^2\frac{dA_{(i-1)}}{da}}{2(i(n'-n)+n)} - 2E_i\right\}.$$

538. In these values of $d\,r$ and $d\,v$, includes all whole numbers, either positive or negative, zero excepted: $d\,r$ and $d\,v$ will now be determined in the latter case, which is very important, because it gives the part of the perturbations that is not periodic.

539. If $i=0$ in the series R in article 449, the only constant term introduced by this value into $d\,r$ will be

$$\frac{m}{2}a^3\left(\frac{dA_0}{da}\right).$$

Again, in finding the integral $d\,a$ the arbitrary constant $a_/$ that ought to have been added, would produce a constant term in $d\,r$. In order to find it, let the origin of the time be at the instant of the conjunction of the two bodies m and m', when

$$n't - nt + \epsilon' - \epsilon = 0\,;$$

whence $\cos 0 = 1$, and the first term of $d\,a$ in article 529 becomes

$$d\,a = -2m'a^2\frac{n}{n'-n}\Sigma A_i\,,$$

whence

$$d\,r = \frac{m'}{2}a^3\frac{dA_0}{da} - 2m'a^2\frac{n}{n'-n}\Sigma A_i\,;$$

where Σ extends to all positive values of i from $i=1$ to $i=\infty$.

540. If these values of $d\,r$ and $d\,a$ be put in equation (148), the result will be

$$d\,v = m'a\left\{\frac{3n}{n'-n}\Sigma A_i - a\left(\frac{dA_0}{da}\right)\right\}\cdot nt\,.$$

And as by article 392 the elliptical parts or r and v that are not periodic, or that do not depend on sines and cosines, are $r = a$, and $v = nt + \in$: those parts of the radius vector and true longitude that are not periodic are expressed by

$$r + dr = a - 2m'a^2 \frac{n}{n' - n} \cdot \sum A_i + \tfrac{1}{2} m'a^3 \left(\frac{dA_0}{da}\right)$$
$$v + dv = nt + \in + m'a \left\{ \frac{3n}{n' - n} \sum A_i - a\left(\frac{dA_0}{da}\right)\right\} nt \qquad \textbf{(150)}$$

in the real orbit.

Thus the perturbations in longitude seem to contain a term that increases indefinitely with the time; were that really the case, the stability of the solar system would soon be at an end. This term however is only introduced by integration, since the differential equations of the perturbations contain no such terms; it is therefore foreign to their nature, and may be made to vanish by a suitable determination of the arbitrary constant quantities. In fact the true longitude of a planet in its disturbed orbit consists of three parts,–of the mean motion, of the equation of the centre, and of the perturbations. The mean motion of the planet is the only quantity in the problem of three bodies that increases with the time: the equation of the centre is a periodic correction which is zero in the apsides and at its maximum in quadratures; and the perturbations being functions of the sines of the mean longitudes of the disturbed and disturbing bodies are consequently periodic, and are applied as corrections to the equation of the centre. All the coefficients of these quantities are functions of the elements of the orbits, which vary periodically but in immensely long periods. The arbitrary constant quantities introduced by integration, must therefore be determined so that the mean motion of the troubled planet may be entirely contained in that part of the longitude represented by v.

541. The values of a, n, e, $\in$, and v, in the preceding equations, are for the epoch $t = 0$, and would be the elliptical values of the elements of the orbit of m, if at that instant the disturbing forces were to cease. Let $n_/ t$ be the mean motion of m given by observation, then the second of the equations under consideration gives

$$n_/ = n\left\{1 + m'a\left(\frac{3n}{n' - n} \cdot \sum A_i - a\left(\frac{dA_0}{da}\right)\right)\right\},$$

and let $a_/$ be the mean distance corresponding to $n_/$ resulting from the equation,

$$n_/^{\,2} = \frac{S + m}{a_/^{\,3}}.$$

If in this last expression $n + n_/ - n$, and $a + a_/ - a$, be put for $n_/$ and $a_/$, and if $\left(n_/ - n\right)^2$, $\left(a_/ - a\right)^2$, which are very small be omitted, then

$$2n\left(n_{/}-n\right)=-\frac{3n^{2}}{a}\left(a_{/}-a\right);$$

and substituting for $n_{/}$ it becomes

$$a-a_{/}=\frac{2m'a^{2}}{3}\left\{\frac{3n}{n'-n}\sum A_{i}-a\left(\frac{dA_{0}}{da}\right)\right\};$$

and as a may be put for $a_{/}$ in the terms multiplied by m', the equations (150) become

$$\begin{aligned} r+d\,r &= a_{/}-\tfrac{1}{6}m'a_{/}{}^{3}\left(\frac{dA_{0}}{da_{/}}\right) \\ v+d\,v &= n_{/}t+\in. \end{aligned}$$

Thus $d\,v$ no longer contains a term proportional to the time, and the mean motion of the disturbedplanet is altogether included in the part of the longitude expressed by v, in consequence of the introduction of the arbitrary constant quantities $n_{/}$ and $a_{/}$, instead of n and a.

The part of $d\,r$ depending on the first powers of the eccentricities may be found by making $i=0$ in the values of $d\,a$, $d\,e$, &c., in article 529; after which their substitution in $d\,r$ of article 536, will give[4]

$$\begin{aligned} d\,r=&-\frac{m'}{4}ae\left\{3a\left(\frac{dA_{0}}{da}\right)+\tfrac{1}{2}a^{2}\left(\frac{d^{2}A_{0}}{da^{2}}\right)\right\}\cos\left(nt+\in-v\right) \\ &-\frac{m'}{4}ae'\left\{3A_{1}-3a\left(\frac{dA_{1}}{da}\right)-\tfrac{1}{2}a^{2}\left(\frac{d^{2}A_{1}}{da^{2}}\right)\right\}\cos\left(nt+\in-v'\right). \end{aligned}$$

The corresponding part of $d\,v$ from article 535 is

$$\begin{aligned} d\,v=&\frac{m'}{2}ae\left\{3a\left(\frac{dA_{0}}{da}\right)+\tfrac{1}{2}a^{2}\left(\frac{d^{2}A_{0}}{da^{2}}\right)\right\}\sin\left(nt+\in-v\right) \\ &+\frac{m'}{2}ae'\left\{2A_{1}-2a\left(\frac{dA_{1}}{da}\right)-\tfrac{1}{2}a^{2}\left(\frac{d^{2}A_{1}}{da^{2}}\right)\right\}\sin\left(nt+\in-v'\right). \end{aligned}$$

542. If the different parts of the value of $d\,r$ and $d\,v$ be added, and if

$$f=\tfrac{1}{4}\left\{3a^{2}\left(\frac{dA_{0}}{da}\right)+\tfrac{1}{2}a^{3}\left(\frac{d^{2}A_{0}}{da^{2}}\right)\right\}$$

$$f' = \tfrac{1}{4}\left\{3aA_1 - 3a^2\left(\frac{dA_1}{da}\right) - \tfrac{1}{2}a^3\left(\frac{d^2A_1}{da^2}\right)\right\}$$

$$f'' = \tfrac{1}{4}\left\{2A_1 - 2a\left(\frac{dA_1}{da}\right) - \tfrac{1}{2}a^2\left(\frac{d^2A_1}{da^2}\right)\right\}$$

The periodic inequalities in the radius vector and true longitude of m when troubled by m', are

$$\begin{aligned}\frac{\delta r}{a} = &-\frac{m'}{6}a_{/}^3\left(\frac{dA_0}{da_{/}}\right) + \frac{m'}{2}\cdot\Sigma.C_i\,.\cos i\left(n't - nt + \epsilon' - \epsilon\right)\\ &- m'.e\,.f\,.\cos\left(nt + \epsilon - \varpi\right) - m'e'f'.\cos\left(nt + \epsilon - \varpi'\right)\\ &+ m'.e.\Sigma.D_i\,.\cos\left\{i\left(n't - nt + \epsilon' - \epsilon\right) + nt + \epsilon - \varpi\right\}\\ &+ m'.e'.\Sigma.E_i\,.\cos\left\{i\left(n't - nt + \epsilon' - \epsilon\right) + nt + \epsilon - \varpi'\right\}\end{aligned}$$

[and]

$$\begin{aligned}\delta v = &+\frac{m'}{2}\cdot\Sigma.F_i\,.\sin i\left(n't - nt + \epsilon' - \epsilon\right)\\ &+ 2m'.e\,.f\,.\sin\left(nt + \epsilon - \varpi\right) + 2m'e'f''.\sin\left(nt + \epsilon - \varpi'\right)\\ &+ m'.e.\Sigma.G_i\,.\sin\left\{i\left(n't - nt + \epsilon' - \epsilon\right) + nt + \epsilon - \varpi\right\}\\ &+ m'.e'.\Sigma.H_i\,.\sin\left\{i\left(n't - nt + \epsilon' - \epsilon\right) + nt + \epsilon - \varpi'\right\}\end{aligned}$$

The action of each disturbing body will produce a similar effect on the radius vector and longitude of *m*, and the sum of all will be perturbations in these two co-ordinates arising from the disturbing action of the whole system on the planet *m*.

543. It has been already observed that each of the periodic variations δa, δe, &c., ought to contain an arbitrary constant quantity $a_{/}$, $e_{/}$, $\varpi_{/}$, &c., introduced by their integrations, so that their true values are

$$a_{/} + \delta a;\ e_{/} + \delta e;\ \varpi_{/} + \delta w;\ \text{\&c. \&c.}$$

Now, if the values of δr, δv, are to express the effects of the disturbing forces on the radius vector and longitude during a given time, these constant quantities must be so determined, that when $t = 0$, they must give

$$e_{/} + \delta e = 0;\ \varpi_{/} + \delta w = 0;\ \text{\&c. \&c.},$$

as was done with δa.

Substituting these values in place of δe, $\delta\varpi$, &c., in equation (149), the resulting values will complete those of δr and δv, which will no longer contain any arbitrary quantity, but will express the whole change in the longitude and distance arising from the action of the disturbing forces. Hence, if (r) (v) be the elliptical values of r and v, given in article 392, but corrected for the secular variation of the elements, the radius vector and longitude of m in its troubled orbit will be determined by

$$r = (r) + d\,r, \quad v = (v) + d\,v\,.$$

Perturbations in Latitude

544. If the second powers of the masses be omitted as well as the squares of the eccentricities, and the products of the eccentricities by the inclination, the orbit at the epoch being the fixed plane, then by article 437

$$d\,s = \frac{d\,z}{a}, \;\; d\,z = y d\,q - x d\,p,$$

and in this case

$$y = a\sin\left(nt + \in\right),\; x = a\cos\left(nt + \in\right),$$

then

$$\frac{d\,z}{a} = d\,q\,.\sin\left(nt + \in\right) - d\,p\,.\cos\left(nt + \in\right),$$

and substituting the values of $d\,q,\;\; d\,p,$ from article 529,

$$\begin{aligned} d\,s = {} & \frac{m'.n^2}{n'^2 - n^2}\cdot\frac{a^2}{a'^2} g\,\sin\left(n't - nt + \in' - \Pi\right) \\ & + \frac{m'^2.n^2.a^2 a'}{2} g\,\Sigma\frac{B_{(i-1)}}{n^2 - \left(n + i\left(n' - n\right)\right)^2}\sin\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - \Pi\right\}. \end{aligned}$$

Now if a plane very little inclined to the orbit of m be assumed for the fixed plane instead of that of the orbit at the epoch, and if $f,\; f',\; q,\; q',$ be the inclinations and longitudes of the nodes of the orbits of m and $m',$ on this new plane; then as g is the tangent of the mutual inclination of the two orbits, and Π the longitude of their mutual intersection, by article 444,

$$g\,\sin\Pi = p' - p; \;\; g\,\cos\Pi = q' - q\,.$$

If these values be substituted in $d\,s,$ and if $(s) + d\,s = s$ be the whole latitude of m in its troubled orbit above the fixed plane, then will

$$s = q\sin(nt+\in) - p\cos(nt+\in)$$
$$+\frac{m'.n^2}{n'^2-n^2}\cdot\frac{a^2}{a'^2}\{(q'-q)\sin(n't+\in')-(p'-p)\sin(n't+\in')\}$$
$$-\frac{m'n^2\ .a^2a'}{2}(q'-q)\Sigma\frac{B_{(i-1)}}{\{i(n'-n)+n\}^2-n^2}\times\sin(i(n't-nt+\in-\in)+nt+\in)$$
$$+\frac{m'n^2\ .a^2a'^2}{2}(p'-p)\Sigma\frac{B_{(i-1)}}{\{i(n'-n)^2+n\}^2-n^2}\times\sin(i(nt-nt+\in-\in)+nt+\in).$$

The two terms independent of m' are the latitude of m above the fixed plane when m remains on the plane of its primitive orbit. If the exact latitude of m be substituted for these two terms, this expression will be more correct.

Each disturbing planet will add an expression to s similar to ds; the sum of the whole will be the true latitude of m when troubled by all the bodies in the system.

545. By a similar process, the perturbations depending on the other powers and products of the eccentricities way be obtained, but it would lead to long and intricate reductions, from which Laplace's method, deduced directly from the equations (87), is exempt.

Notes

[1] See note 16, *Preliminary Dissertation.*
[2] See note 4, *Introduction.*
[3] The third term element $\in'$ *reads* $\in$` in the 1st edition.
[4] The term $-3a\left(\frac{dA_1}{da}\right)$ *reads* $-3a\left(\frac{dA_/}{da}\right)$ in the 1st edition.