BOOK II

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CHAPTER IX

SECOND METHOD OF FINDING THE PERTURBATIONS OF A PLANET IN LONGITUDE, LATITUDE, AND DISTANCE

Determination of the general Equations

546. TO determine the perturbations dv, dr, ds, from the three general equations,[1]

$$\frac{d^2x}{dt^2}+\frac{mx}{r^3}=\frac{dR}{dx}$$
$$\frac{d^2y}{dt^2}+\frac{my}{r^3}=\frac{dR}{dy}$$
$$\frac{d^2z}{dt^2}+\frac{mz}{r^3}=\frac{dR}{dz}.$$

The sum of these equations respectively multiplied by dx, dy, dz is

$$\frac{dxd^2x+dyd^2y+dzd^2z}{dt^2}+\frac{m\left(xdx+ydy+zdz\right)}{r^3}=dx\left(\frac{dR}{dx}\right)+dy\left(\frac{dR}{dy}\right)+dz\left(\frac{dR}{dz}\right). \quad \textbf{(151)}$$

The integral of which is evidently

$$\frac{dx^2+dy^2+dz^2}{dt^2}-\frac{2m}{r}+\frac{m}{a}=2\int \mathrm{d}R\,. \quad \textbf{(152)}$$

The differential of R is only relative to the co-ordinates of m, because the motions of that body alone are under consideration; a is an arbitrary constant quantity introduced by integration; it is half the greater axis of the orbit of m when R is zero. Again, the same equations, respectively multiplied by x, y, and z, and added to the preceding integral, give

$$\frac{xd^2x+yd^2y+zd^2z}{dt^2}+\frac{dx^2+dy^2+dz^2}{dt^2}+\frac{m\left(x^2+y^2+z^2\right)}{r^3}$$
$$-\frac{2m}{r}+\frac{m}{a}=x\left(\frac{dR}{dx}\right)+y\left(\frac{dR}{dy}\right)+z\left(\frac{dR}{dz}\right)+2\int \mathrm{d}R\,.$$

The two first members of this equation are equal to $\frac{1}{2}\frac{d^2r^2}{dt^2}$, the third is $\frac{m}{r}$ and if to abridge rR' be put for

$$x\left(\frac{dR}{dx}\right)+y\left(\frac{dR}{dy}\right)+z\left(\frac{dR}{dz}\right),$$

the equation becomes

$$\frac{1}{2}\frac{d^2r^2}{dt^2}-\frac{m}{r}+\frac{m}{a}=2\int \mathrm{d}\,.R+rR'.$$

Let dv be the indefinitely small angle mSh, fig. 89, contained between

$$Sm=r,\text{ and } Sh=r+dr,\text{ then } mh^2=ma^2+ah^2;$$

but

$$ma=rdv,\text{ and } ah=dr,$$

hence

$$mh^2=dr^2+r^2dv^2=dx^2+dy^2+dz^2.$$

But

$$xd^2x+yd^2y+zd^2z=d\left(xdx+ydy+zdz\right)-\left(dx^2+dy^2+dz^2\right)=rd^2r-r^2dv^2;$$

so that the equation in question becomes,

$$\frac{rd^2r-r^2\,.dv^2}{dt^2}+\frac{m}{r}=rR',$$

whence[2]

$$\frac{dv^2}{dt^2}-\frac{d^2r}{rdt^2}-\frac{m}{r^3}=-\frac{1}{r}R'.$$

547. In solving equations by approximation, a value of the unknown quantity is found by omitting some of the smaller terms, then the value so found is substituted in the equation, and a new value is sought, including the terms that were at first omitted.

Now values of r and v have been determined in the elliptical orbit by omitting the parts containing the disturbing forces, but if $r+\delta r$ [and] $v+\delta v$ be put for r and v in the preceding equations, the parts containing the elliptical motion may be omitted, and the remaining terms will give the perturbations. It is evident, however, that this substitution must not be made in R, since it contains the first powers of the disturbing action already. Consequently, the equations in question become

$$\frac{d^2\,.r\delta r}{dt^2}+\frac{mr\delta r}{r^3}=2\int \mathrm{d}R+rR'$$

$$\frac{2r^2dv\,.\,d\delta v}{dt^2}=\frac{rd^2\delta r-\delta r\,.d^2r}{dt^2}-\frac{3mr\delta r}{r^3}-rR'$$

and eliminating $\dfrac{mrdr}{r^3}$ from the second by means of the first

$$ddv = \frac{d\left\{dr\,.\,d\;r + 2r\,.\,ddr\right\} - dt^2\left(3\int \mathrm{d}R + 2rR'\right)}{r^2 dv}. \qquad \textbf{(153)}$$

548. The integral of the angle $dv = mSh$ does not lie all in one plane, but it is easy to obtain a value of that indefinitely small angle in functions of its projection $pS\mathrm{B} = dv'$. For let $p\mathrm{B}$ be the projection of the arc mh on the fixed plane of the ecliptic, then $Sm = r$, $mp = s$, the tangent of the latitude, and the curtate distance $Sp = \dfrac{r}{\sqrt{1+s^2}}$.

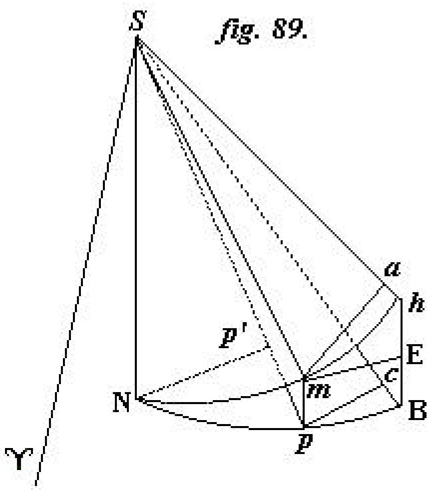

fig. 89.

Draw $m\mathrm{E}$ at right angles to Bh; and describe the arc pc with the radius Sp. Now the arc mh being indefinitely small, its projection $p\mathrm{B}$ is indefinitely small, therefore both may be taken in place of their sines; also[3]

$$(mh)^2 = (m\mathrm{E})^2 + (\mathrm{E}h)^2$$

and as

$$m\mathrm{E} = p\mathrm{B},$$

therefore[4]

$$(pc)^2 + (c\mathrm{B})^2 + (\mathrm{E}h)^2 = (mh)^2.$$

But

$$pc = \frac{rdv'}{\sqrt{1+s^2}};\; c\mathrm{B} = d\,.\,\frac{r}{\sqrt{1+s^2}} = \frac{dr\left(1+s^2\right) - rsds}{\left(1+s^2\right)^{\frac{3}{2}}}.$$

Again

$$mp = \frac{sr}{\sqrt{1+s^2}}.$$

And[5]

$$h\mathrm{E} = d\,.\,(mp) = \frac{rds + sdr\left(1+s^2\right)}{\left(1+s^2\right)^{\frac{3}{2}}};$$

hence

$$(mh)^2 = \frac{r^2 dv'^2}{1+s^2} + dr^2 + \frac{r^2 ds^2}{\left(1+s^2\right)^2};$$

but[6]

$$(mh)^2 = r^2 dv^2 + dr^2,$$

and lastly,

$$dv' = dv \cdot \frac{\sqrt{\left(1+s^2\right)^2 - \frac{ds^2}{dv^2}}}{\sqrt{1+s^2}}. \qquad \textbf{(154)}$$

Thus dv' is known when ds is determined; however, if the latitude be estimated from a known position of the orbit of the planet itself at any given epoch, instead of from the fixed plane of the ecliptic, it will be zero at that instant; and any latitude the planet may have at a subsequent period, can only arise from the disturbing forces, and must on that account be very small.

549. Assuming therefore *NpB*, fig. 89, to have been the orbit of the planet *m* at any given time, *s* and *ds* will be so small, that their squares may be omitted, and then $dv = dv'$. Also the radius vector *r*, only differs from the curtate distance $\frac{r}{\sqrt{1+s^2}}$ by the extremely small quantity $\frac{1}{2}rs^2$ which may be omitted, and then the true longitude v may be estimated on the plane *NpB* without sensible error; so that

$$SN = x = r\cos v; \quad Np' = y = r\sin v;$$

and as $z\left(\frac{dR}{dz}\right)$ is so small that it may be omitted,

$$rR' = x\left(\frac{dR}{dx}\right) + y\left(\frac{dR}{dy}\right) = r\left(\frac{dR}{dr}\right);$$

hence, the equation which determines the perturbations in the radius vector becomes[7]

$$\frac{d^2 . rdr}{dt^2} + \frac{mrdr}{r^3} = 2\int \mathrm{d}R + r\left(\frac{dR}{dr}\right). \qquad \textbf{(155)}$$

550. It was shown in article 372, that r^2dv is the area described by the body in the indefinitely small time dt, therefore

$$r^2dv = \sqrt{ma\left(1-e^2\right)}dt = na^2\sqrt{1-e^2}\,.\,dt$$

hence the value of ddv becomes

$$ddv = \frac{1}{\sqrt{1-e^2}}\left\{\frac{d\left(2rd\,.dr + dr\,.dr\right)}{a^2ndt^2} - \frac{an}{m}\left(3\int \mathrm{d}R + 2r\left(\frac{dR}{dr}\right)\right)\right\}$$

and its integral is

$$d v = \frac{1}{\sqrt{1-e^2}} \left\{ \frac{2rd\,.d\,r + dr\,.d\,r}{a^2ndt} - \frac{an}{m}\left(3\iint dt\,.\,\mathrm{d}R + 2\int r\left(\frac{dR}{dr}\right)dt\right)\right\} \qquad \textbf{(156)}$$

which determines the perturbations of m in longitude.

551. Since the orbit of m at the epoch is taken as the fixed plane, the only latitude the planet will have at a subsequent period must arise from the perturbations, and may therefore be represented by $d\,s$; hence $z = rd\,s$. And substituting this value of z in the third of the equations of motion in article 546, it becomes

$$\frac{d^2\,.\,rd\,s}{dt^2} + \frac{m\,.\,rd\,s}{r^3} - \frac{dR}{dz} = 0\,. \qquad \textbf{(157)}$$

A value of $d\,s$ may easily be found from this; and if it be then desired to refer the position of the planet to a plane which is but little inclined to that of its primitive orbit, it will only be necessary to add to this value of $d\,s$ the latitude of the planet, supposing it not to quit the plane of its primitive orbit.

Perturbations in the Radius Vector

552. These are obtained by successive approximations from the equation

$$\frac{d^2rdr}{dt^2} + \frac{mrd\,r}{r^3} = 2\int \mathrm{d}R + r\left(\frac{dR}{dr}\right).$$

A value of $d\,r$ is first determined by omitting the eccentricities; that value is substituted in the same equation, and a new value of $d\,r$ is found, including the first powers of the eccentricities; that is again substituted, and a third value of $d\,r$ is obtained, containing the squares and products of the eccentricities and inclinations, and so on, till the remaining or rejected quantities are less than the errors of observation.

553. Supposing the orbits to be circular, then $r^{-3} = a^{-3}$; and by article 383, $\frac{m}{a^3} = n^2$. And if the mass of the planet be omitted when compared with that of the sun taken as the unit, the preceding equation, after these substitutions, becomes

$$\frac{d^2\,.\,rd\,r}{dt^2} + n^2rdr = 2\int \mathrm{d}R + r\left(\frac{dR}{dr}\right).$$

But

$$r\left(\frac{dR}{dr}\right) = a\left(\frac{dR}{du}\right);$$

and in this case

$$R = \frac{m'}{2} \sum A_i \cos i \left(n't - nt + \in' - \in \right).$$

When

$$i = 0, \ \cos i \left(n't - nt + \in' - \in \right) = 1,$$

$$R = \frac{m}{2} A_0 + \frac{m}{2} \cdot \sum . A_i \cos i \left(n't - nt + \in' - \in \right),$$

and as dR is the differential of R with regard to nt alone, therefore

$$2 \int \mathrm{d}R + r \left(\frac{dR}{dr} \right) = 2m' g + \frac{m'}{2} a \left(\frac{dA_0}{da} \right) + \frac{m'}{2} \sum \left\{ \frac{2n}{n - n'} A_i + a \left(\frac{dA_i}{da} \right) \right\} \times \cos i \left(n't - nt + \in' - \in \right);$$

whence

$$\frac{d^2 r d r}{dt^2} + n^2 r d r = 2m' g + \frac{m'}{2} a . \left(\frac{dA_0}{da} \right) + \frac{m'}{2} \sum \left\{ \frac{2n}{n - n'} A_i + a \left(\frac{dA_i}{da} \right) \times \cos i \left(n't - nt + \in' - \in \right) \right\}.$$

The integral of this equation is

$$\frac{r d r}{a^2} = B + B' . \cos i \left(n't - nt + \in' - \in \right),$$

B and B' being indeterminate coefficients, then

$$\frac{d^2 r d r}{dt^2} + n^2 r d r = B n^2 a^2 + B' a^2 \left(n^2 - i^2 \left(n' - n \right)^2 \right) \cos i \left(n't - nt + \in' - \in \right).$$

And comparing the coefficients of like cosines,

$$B = 2m' a g + \frac{m'}{2} a^2 \left(\frac{dA_0}{da} \right),$$

$$B' = \frac{m'}{2} \frac{a n^2}{n^2 - i^2 \left(n - n' \right)^2} \cdot \sum \cdot \left\{ \frac{2n}{n - n'} A_i + a \left(\frac{dA_i}{da} \right) \right\},$$

and so

$$\frac{r d r}{a^2} = 2m' a g + \frac{m'}{2} a^2 \left(\frac{dA_0}{da} \right) + \frac{m' n^2}{2} \cdot \sum \cdot \frac{\left\{ \frac{2n}{n - n'} a A_i + a^2 \left(\frac{dA_i}{da} \right) \right\}}{n^2 - i^2 \left(n - n' \right)^2} \times \cos i \left(n't - nt + \in' - \in \right);$$

or if a be put for r in the first member, and because by article 536,

$$C_i = \frac{n^2\left\{\frac{2n}{n-n'}aA_i + a^2\left(\frac{dA_i}{da}\right)\right\}}{n^2 - i^2(n-n')^2}$$

[then]

$$\frac{d\,r}{a} = 2m'a\,g + \frac{m'}{2}a^2\left(\frac{dA_0}{da}\right) + \frac{m'}{2}\Sigma C_i \cos i\left(n't - nt + \in' - \in\right);$$

which is the first approximation.

554. When the first powers of the eccentricities are included,

$$r^{-3} = \frac{1}{a^3}\left\{1 + 3e\cos\left(nt + \in - v\right)\right\};$$

and therefore

$$\frac{d^2\,r d\,r}{dt^2} + n^2\,r d\,r + 3n^2 a\,.\,d\,r\,.\,e\cos\left(nt + \in - v\right) = 2\int \mathrm{d}R + r\left(\frac{dR}{dr}\right),$$

but

$$R = \frac{m'}{2}\Sigma M_0 e\cos\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v\right\} + \frac{m'}{2}\Sigma M_1 e'\cos\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v'\right\}$$

therefore

$$2\int \mathrm{d}R + r\left(\frac{dR}{dr}\right) = \frac{m'}{2}\Sigma\left\{\frac{2(i-1)n}{i(n-n')-n}M_0 + a\left(\frac{dM_0}{da}\right)\right\}e\cos\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v\right\}$$

$$+ \frac{m'}{2}\Sigma\left\{\frac{2(i-1)n}{i(n-n')-n}M_1 + a\left(\frac{dM_1}{da}\right)\right\}e'\cos\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v'\right\}.$$

By the substitution of this quantity, and of the preceding value of $\frac{d\,r}{a}$,

$$\frac{d^2\,r d\,r}{dt^2} + n^2\,r d\,r = -\frac{m'}{2}\Sigma\left\{3a^2n^2C_i - \frac{2(i-1)n}{i(n-n')-n}M_0 - a\frac{dM_0}{da}\right\}e\cos\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v\right\}$$

$$+ \frac{m'}{2}\Sigma\left\{\frac{2(i-1)n}{i(n-n')-n}M_1 + a\frac{dM_1}{da}\right\}e'\cos\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v'\right\}.$$

Let

$$\frac{r d\,r}{a^2} = \frac{m'}{2}Be\cos\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v\right\} + \frac{m'}{2}B'e'\cos\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - v'\right\}$$

B and B' being indeterminate coefficients, then

$$\frac{d^2 r\delta r}{dt^2}+n^2 r\delta r=+\frac{m'}{2}\cdot Ba^2\left\{n^2-\left(i\left(n-n'\right)+n\right)^2\right\}.e.\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in-\nu\right\}$$
$$+\frac{m'}{2}\cdot B'a^2\left\{n^2-\left(i\left(n-n'\right)+n\right)^2\right\}.e'.\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in-\nu'\right\}.$$

If to abridge

$$K_i=3C_i-\frac{2\left(i-1\right)n}{i\left(n-n'\right)-n}aM_0-a^2\cdot\frac{dM_0}{da}$$
$$L_i=-\frac{2\left(i-1\right)}{i\left(n-n'\right)-n}aM_1-a^2\frac{dM_1}{da},$$
$$B=\frac{-n^2.K_i}{n^2-\left\{i\left(n-n'\right)+n\right\}^2};\quad B'=\frac{-n^2.L_i}{n^2-\left\{i\left(n-n'\right)+n\right\}^2};$$

and because $a^3n^2=1$,

$$\frac{r\delta r}{a^2}=\Sigma\frac{m'n^2}{2\left\{i\left(n-n'\right)+n\right\}^2-2n^2}\begin{Bmatrix}+K_i.e\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in-\nu\right\}\\+L_i.e'\cos\left\{i\left(n't-nt+\in'-\in\right)+nt+\in-\nu'\right\}\end{Bmatrix};$$

where i may have any whole value, positive or negative, except zero. But in order to have the complete value of $\frac{r\delta r}{a^2}$, according to the theory of linear equations the integral of

$$\frac{d^2.r\delta r}{dt^2}+n^2 r\delta r=0$$

must be added. The true integral of this equation is

$$\frac{r\delta r}{a^2}=m'fe.\cos\left(nt\left(1+c\right)+\in-\nu\right)+m'f'e'.\cos\left(nt\left(1+c'\right)+\in-\nu'\right)$$

where c and c' are given functions of the elements; but if it be assumed as is generally done, that the elliptical elements have already been corrected by their secular variations c and c' may be omitted, and then

$$\frac{r\delta r}{a^2}=m'fe.\cos\left(nt+\in-\nu\right)+m'f'e'.\cos\left(nt+\in-\nu'\right).$$

If all the parts of $\frac{r\delta r}{a^2}$ that have been determined in this and in the first approximation be collected, and if [8] $a\left(1-\cos\left(nt+\in-\nu\right)\right)$ be put for r, then will

$$\frac{dr}{a} = +2m'ag + \frac{m'}{2}a^2\left(\frac{dA_0}{da}\right)$$
$$+m'f.e.\cos(nt+\in-v)+m'f'.e'.\cos(nt+\in-v')$$
$$+\frac{m'}{2}\cdot\frac{n^2}{\left(n^2-i^2(n'-n)^2\right)}\cdot\Sigma\left\{\frac{2n}{n-n'}aA_i+a^2\left(\frac{dA_i}{da}\right)\right\}\cos(n't-nt+\in'-\in)$$
$$+\frac{m'}{2}\cdot\Sigma\cdot\left\{C_i+\frac{n^2K_i}{\{i(n'-n)+n\}^2-n^2}\right\}\cdot e\cdot\cos\{i(n't-nt+\in'-\in)+nt+\in-v\}$$
$$+\frac{m'}{2}\cdot\Sigma\cdot\left\{\frac{nL_i}{\{i(n'-n)+n\}^2-n^2}\right\}\cdot e'\cdot\cos\{i(n't-nt+\in'-\in)+nt+\in-v\}.$$

If substitution be made for K_i and L_i, it will be found that the coefficients in this expression are identical with those in article 536, so that

$$\frac{n^2\,\Sigma\left\{\frac{2n}{n-n'}aA_i+a^2\left(\frac{dA_i}{da}\right)\right\}}{n^2-i^2(n'-n)^2}=C_i,$$
$$C_i+\frac{n^2K_i}{\{i(n'-n)+n\}-n^2}=D_i,$$
$$\frac{n^2L_i}{\{i(n'-n)+n\}^2-n^2}=F_i,$$

consequently

$$\frac{dr}{a}=+2m'ag+\frac{m'}{2}\cdot a^2\left(\frac{dA_0}{da}\right)+\frac{m'}{2}\Sigma C_i\cos i(n't-nt+\in'-\in)$$
$$+m'.\,fe.\cos(nt+\in-v)+m'.\,f'.\,e'.\cos(nt+\in-v)$$
$$+m'.\Sigma.D_i.e.\cos\{i(n't-nt+\in'-\in)+nt+\in-v\}$$
$$+m'.\Sigma.F_i.e'.\cos\{i(n't-nt+\in'-\in)+nt+\in-v'\}.$$

Perturbations in Longitude

555. The perturbations in longitude may now he found from equation (156); which becomes, when e^2 is omitted, and $m=a^3n^2=1$,

$$dv=\frac{2rd.dr+dr.dr}{a^2.ndt}-3an\iint dtdR-2an\int r\left(\frac{dR}{dr}\right)dt.$$

By the substitution of the preceding values of R and δr it will be found that the perturbations in longitude are[9]

$$\delta v = -m'a\left(3g + a\left(\frac{dA_0}{da}\right)\right).nt$$
$$+\frac{m'}{2}\sum\left\{-\frac{n^2}{i\left(n-n'\right)^2}aA_i + \frac{2n^3\left\{\frac{2n}{n-n'}aA_i + a^2\left(\frac{dA_i}{da}\right)\right\}}{i\left(n-n'\right)n^2 - i^2\left(n-n'\right)^2}\right\}\times\sin i\left(n't-nt+\epsilon'-\epsilon\right)$$
$$+m'f_{/}e\sin\left(nt+\epsilon-v\right)+m'f'_{/}e'.\sin\left(nt+\epsilon-v'\right)$$
$$+m'e\sum G_i\sin\left\{i\left(nt-nt+\epsilon-v\right)+nt+\epsilon-v\right\}$$
$$+m'e'\sum H_i\sin\left\{i\left(nt-nt+\epsilon-v\right)+nt+\epsilon-v'\right\}+C,$$

where

$$f_{/} = 3a^2\frac{dA_0}{da}+a^3\frac{d^2A_0}{da^2}+2ag$$

[and]

$$f'_{/} = \tfrac{3}{2}aA_1-\tfrac{3}{2}a^2\frac{dA_1}{da}-a^3\frac{d^2A_1}{da^2}-2f'.$$

556. If all the periodic terms be omitted in the expressions $r+\delta r$ and $v+\delta v$, they become[10]

$$r+\delta r = a+2m'a^2g+\tfrac{1}{2}m'a^2\left(\frac{dA_0}{da}\right)$$
$$v+\delta v = nt+\epsilon-m'\left(3ag+a^2\left(\frac{dA_0}{da}\right)\right).nt;$$

$v+\delta v$ is the mean longitude of the planet at the end of the time t; and if it be assumed that this longitude is the same as the elliptical orbit of the planet, and in the orbit it really describes, this condition will determine g. Whence

$$g = -\tfrac{1}{3}a\left(\frac{dA_0}{da}\right);$$

and, as before

$$r+\delta r = a-\frac{m'}{6}a^3\left(\frac{dA_0}{da}\right),$$

which is the constant part of the radius vector in the troubled orbit.

Thus a is not the mean distance of the planet from the sun in the troubled orbit, as it is in the elliptical orbit. In the latter case a is deduced from the mean motion by the equation

$$n^2 = \frac{1}{a^3},$$

whereas in the troubled orbit it is

$$a - \frac{m'}{6} a^3 \left(\frac{dA_0}{da} \right).$$

Therefore the mean motion and periodic time are different from what they would have been had there been no disturbance; but when they are produced they are permanent, and unchangeable in their quantity by the subsequent action of the other bodies of the system.

The perturbations in the co-ordinates of a planet depend on the angular distances of the disturbed and disturbing bodies, that is, on the differences of their mean longitudes; but terms of the form

$$f_{/}\, e \sin\left(nt + \in - v'\right), \; f'_{/} e' \sin\left(nt + \in - v'\right)$$

belong to elliptical motion; they form a part of the equation of the centre, but they will vanish from dv if $f_{/}$ and $f'_{/}$, which are perfectly arbitrary, be made zero; in that case

$$f = \tfrac{1}{2}\left(\tfrac{7}{3} a^2 \frac{dA_0}{da} + a^3 \frac{d^2 A_0}{da^2} \right);$$

$$f' = \tfrac{1}{2}\left\{ \tfrac{3}{2} a A_1 - \tfrac{3}{2} a^2 \frac{dA_1}{da} - a^3 \frac{d^2 A_1}{da} \right\}.$$

and as the arbitrary constant quantity C may be made zero, the perturbations in the radius vector and longitude of m are

$$\begin{aligned} dr = & -\frac{m'}{6} a^2 \left(\frac{dA_0}{da} \right) + \frac{m'}{2} \sum C_i \cos i\left(n't - nt + \in' - \in\right) \\ & + m' f e \cos\left(nt + \in - v\right) + m' f' e' \cos\left(nt + \in - v'\right) \\ & + m' e \sum D_i \cos\left\{ i\left(n't - nt + \in' - \in\right) + nt + \in - v \right\} \\ & + m' e' \sum E_i \cos\left\{ i\left(n't - nt + \in' - \in\right) + nt + \in - v' \right\}; \end{aligned} \qquad \textbf{(158)}$$

[and]

$$\begin{aligned} dv = & + \frac{m'}{2} \sum F_i \sin i\left(n't - nt + \in' - \in\right) \\ & + m' e \sum G_i \sin\left\{ i\left(n't - nt + \in' - \in\right) + nt + \in - v \right\} \\ & + m' e' \sum H_i \sin\left\{ i\left(n't - nt + \in' - \in\right) + nt + \in - v' \right\}. \end{aligned} \qquad \textbf{(159)}$$

The coefficients being the same as in article 537, and i may have any whole value, except zero.

557. The integral

$$\frac{rdr}{a^2} = m'.f\,.e.\cos\left(nt(1+c)+\in-v\right)+m'.f'\,.e'.\cos\left(nt(1+c')+\in-v\right),$$

by the resolution of the cosines becomes

$$\frac{rdr}{a^2} = +m'.f\,.e.\cos\left(nt+\in-v\right)+m'.f'.e'.\cos\left(nt+\in-v'\right)$$
$$-m'.f\,.e.cnt.\sin\left(nt+\in-v\right)-m'.f'.e'.c'nt.\sin\left(nt+\in-v'\right);$$

and as it is given under this form by direct integration, it was very[11] embarrassing to mathematicians, because the terms containing the arcs cnt, $c'nt$, as coefficients, increase indefinitely with the time, and if such inequalities really had existence in our system, its stability would soon be at end. The expression (98) for the radius vector does not contain a term that increases with the time, neither does the series R; consequently the arc nt could not be introduced into the differential equation (155), unless R contained terms of the form

$$A.\frac{\sin}{\cos}\left(a+bt+gt^2+\&\text{c.}\right),$$

the differential of which would produce them.

Now, the powers and products of sines and cosines introduce the sines and cosines of multiple arcs, but never the sines or cosines of the powers of arcs; consequently R does not contain terms of the preceding form, and therefore the differential equation (155) does not contain any term that increases with the time. Terms in the finite equations, that have the arc nt as coefficient, really arise from the imperfection of analysis, by which, in the course of integration, periodic terms, such as $A.\cos\left(nt+\in-v\right)$, are introduced under their developed form $a+bt+gt^2+\&\text{c.}$; and, as Mr. Herschel[12] observes, that is not done at once, but by degrees; a first approximation giving only a, the next bt, and so on. In stopping here, it is obvious that we should mistake the nature of this inequality, and that a really periodical function, from the effect of an imperfect approximation, would appear under the form of one not periodical, and would lead to erroneous conclusions as to the stability of the system and the general laws of its perturbations.

When, by this manner of integration, terms that increase with the time are introduced, the method of reducing the integrals to the periodic form will be found in a Memoir by Laplace, in the *Mem. Acad. Sci.*, 1772, and in the fifth chapter, second book, of the *Mécanique Céleste.*[13]

Perturbations in Latitude

558. Those are found by substituting

$$\frac{dR}{dz} = -\frac{m'}{a'^2}g\ \sin\left(n't + \in' - \Pi\right) + \frac{m'}{2}a'\sum B_{(i-1)}g\ \sin\left\{i\left(n't - nt + \in' - \in\right) + nt + \in - \Pi\right\}$$

in

$$\frac{d^2 r\delta s}{dt^2} + n^2 . r\delta s = \frac{dR}{dz};$$

where the primitive orbit of m is assumed to be the fixed plane, and the product of the eccentricity by the inclination neglected. Making $a^2 n^2 = 1$, and integrating, the result is

$$\begin{aligned}\frac{\delta s}{a} = &+ \frac{m' n^2}{n'^2 - n^2}\cdot\frac{a^2}{a'^2}g\ \sin\left(n't + \in' - \Pi\right)\\ &+ \frac{m'}{2}\cdot\frac{a'}{a}g\ \frac{\sum B_{(i-1)}}{n^2 - \left(i\left(n' - n\right) + n\right)^2}\sin\left(i\left(n't - nt + \in' - \in\right) + nt + \in - \Pi\right).\end{aligned} \qquad \textbf{(160)}$$

This expression is the same with that in article 544. No constant quantities are added, because, being arbitrary, they are assumed to be zero, which does not interfere with the generality of the problem, and is more convenient for use: i may have any whole value, positive or negative, zero excepted.

Perturbations, including the Squares of the Eccentricities and Inclinations

559. When the approximation extends to the squares and products of the eccentricities and inclinations

$$r^{-3} = \frac{1}{a^3}\left\{1 + 3e\cos\left(nt + \in - v\right) + 3e^2\cos 2\left(nt + \in - v\right)\right\};$$

and by article 451,

$$\begin{aligned}R = &+ \frac{m'}{2}\cdot\sum N . \cos\left\{i\left(n't - nt + \in' - \in\right) + 2nt + 2\in + L\right\}\\ &+ \frac{m'}{2}\cdot\sum N' . \cos\left\{i\left(n't - nt + \in' - \in\right) + L'\right\};\end{aligned}$$

whence

$$\begin{aligned}&2\int \mathrm{d}R + r\left(\frac{dR}{dr}\right) =\\ &+ \frac{m'}{2}\cdot\sum\left\{a\frac{dN}{da} + N\cdot\frac{2\left(2-i\right)n}{i\left(n' - n\right) + 2n}\right\}\cos\left\{i\left(n't - nt + \in' - \in\right) + 2nt + 2\in + L\right\}\\ &+ \frac{m'}{2}\cdot\sum\left\{a\frac{dN'}{da} - N'\cdot\frac{2n}{\left(n' - n\right)}\right\}\cos\left\{i\left(n't - nt + \in' - \in\right) + L'\right\}\end{aligned}$$

hence

$$\frac{d^2 r\delta r}{dt^2} + n^2 r\delta r + 3n^2 a . \delta r . \left\{ e\cos\left(nt + \epsilon - \varpi\right) + e^2 \cos 2\left(nt + \epsilon - \varpi\right) \right\} =$$
$$+ \frac{m'}{2} \cdot \Sigma \left\{ a\frac{dN}{da} + N \cdot \frac{2\left(2 - i\right)n}{i\left(n' - n\right) + 2n} \right\} \cos\left\{ i\left(n't - nt + \epsilon' - \epsilon\right) + 2n + L \right\}$$
$$+ \frac{m'}{2} \cdot \Sigma \left\{ a\frac{dN'}{da} - N' \cdot \frac{2n}{\left(n' - n\right)} \right\} \cos\left\{ i\left(n't - nt + \epsilon' - \epsilon\right) + L' \right\}.$$

The value of $\frac{\delta r}{a}$, given in article 558, must be substituted in the last term of the first member; but as all terms are rejected that do not contain the squares or products of the eccentricities and inclinations, the only part of $\frac{\delta r}{a}$ that is requisite is

$$\frac{\delta r}{a} = + \frac{m'}{2} \Sigma . C_i \cos i\left(n't - nt + \epsilon' - \epsilon\right)$$
$$+ m'e \Sigma D_i \cos\left\{ i\left(n't - nt + \epsilon' - \epsilon\right) + nt + \epsilon - \varpi \right\}$$
$$+ m'e' \Sigma E_i \cos\left\{ i\left(n't - nt + \epsilon' - \epsilon\right) + nt + \epsilon - \varpi' \right\};$$

where *i* may have every value, positive or negative, zero excepted. But if *i* be made negative in the two last terms, and if D_i', E_i', be the two coefficients in this case, then

$$\frac{\delta r}{a} = + \frac{m'}{2} \Sigma C_i \cos i\left(n't - nt + \epsilon' - \epsilon\right)$$
$$+ m'e \Sigma D_i \cos\left\{ i\left(n't - nt + \epsilon' - \epsilon\right) + nt + \epsilon - \varpi \right\}$$
$$+ m'e \Sigma D_i' \cos\left\{ -i\left(n't - nt + \epsilon' - \epsilon\right) + nt + \epsilon - \varpi \right\} \qquad \textbf{(161)}$$
$$+ m'e' \Sigma E_i \cos\left\{ i\left(n't - nt + \epsilon' - \epsilon\right) + nt + \epsilon - \varpi' \right\}$$
$$+ m'e' \Sigma E_i' \cos\left\{ -i\left(n't - nt + \epsilon' - \epsilon\right) + nt + \epsilon - \varpi' \right\};$$

but now *i* can only be a *positive* whole number.

When this quantity is substituted in the last term of the first member for δr, and terms of the second order alone retained, the differential equation becomes, when integrated by the method of indeterminate coefficients, or otherwise,

$$\frac{rdr}{a^2} = +\frac{m'.n^2}{\{i(n'-n)+3n\}\cdot\{i(n'-n)+n\}}\times$$

$$\left\{\begin{array}{l} \tfrac{3}{2}e^2.\sum.\{\tfrac{1}{2}C_i+D_i\}.\cos\{i(n't-nt+\epsilon'-\epsilon)+2nt+2\epsilon-2v\} \\ +\tfrac{3}{2}ee'.\sum.E_i.\cos\{i(n't-nt+\epsilon'-\epsilon)+2nt+2\epsilon-v-v'\} \\ -\tfrac{1}{2}\sum\left\{\dfrac{2(2-i)n}{in'+(2-i)n}aN+a^2\dfrac{dN}{da}\right\}\cdot\cos\{i(n't-nt+\epsilon'-\epsilon)+2nt+L\} \end{array}\right\} \qquad \textbf{(162)}$$

$$+\frac{m'.n^2}{\{i(n'-n)-n\}\cdot\{i(n'-n)+n\}}\times$$

$$\left\{\begin{array}{l} \tfrac{3}{2}ee'.\sum.E_i.\cos\{i(n't-nt+\epsilon'-\epsilon)-v'+v\} \\ +\tfrac{3}{2}ee'.\sum.E'_i.\cos\{i(n't-nt+\epsilon'-\epsilon)+v'-v\} \\ +\tfrac{3}{2}\sum\{D_i+D'_i\}.e^2.\cos i(n't-nt+\epsilon'-\epsilon) \\ -\tfrac{1}{2}\sum\left\{a^2\dfrac{dN'}{da}-\dfrac{2n}{n'-n}aN'\right\}\cdot\cos\{i(n't-nt+\epsilon'-\epsilon+L')\} \end{array}\right\}.$$

Now in order to obtain the value of $\frac{dr}{a}$ from this expression it must be observed that

$$\frac{rdr}{a^2} = \frac{r}{a}\cdot\frac{dr}{a};$$

and when the elliptical value of r is substituted it becomes

$$\frac{rdr}{a^2} = \frac{dr}{a}\left\{1+\tfrac{1}{2}e^2 - e.\cos(nt+\epsilon-v) - \tfrac{1}{2}e^2\cos 2(nt+\epsilon-v)\right\}$$

whence

$$\frac{dr}{a} = \frac{rdr}{a^2} - \frac{dr}{a}\left\{\tfrac{1}{2}e^2 - e\cos(nt+\epsilon-v) - \tfrac{1}{2}e^2\cos 2(nt+\epsilon-v)\right\}.$$

If the value of $\frac{dr}{a}$ from equation (161) be substituted in the second member, it will be found, after the reduction of the products of the cosines, that the perturbations in the radius vector depending on the second powers of the eccentricities and inclinations are expressed by

$$\begin{aligned}\frac{dr}{a} &= +\frac{rdr}{a^2}+\frac{m'}{4}\cdot\Sigma\cdot\left\{\tfrac{1}{2}C_i+2D_i\right\}\times e^2\,.\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+2nt+2\epsilon-2\varpi\right\}\\ &+\frac{m'}{2}\cdot\Sigma\cdot E_i ee'\,.\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+2nt+2\epsilon-\varpi-\varpi'\right\}\\ &+\frac{m'}{2}\cdot\Sigma\left\{D_i+D'_i-\tfrac{1}{2}C_i\right\}e^2\cos i\left(n't-nt+\epsilon'-\epsilon\right)\\ &+\frac{m'}{2}\cdot\Sigma\cdot E_i ee'\,.\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+\varpi-\varpi'\right\}\\ &+\frac{m'}{2}\cdot\Sigma\cdot E_i\,.ee'\,.\cos\left\{i\left(n't-nt+\epsilon'-\epsilon\right)-\varpi+\varpi'\right\};\end{aligned} \qquad \textbf{(163)}$$

where $\frac{rdr}{a^2}$ represents equation (162).

560. With the values of $\frac{dr}{a}$ in (163) and (161), together with those terms of R that depend on the second powers and products of the eccentricities and inclinations, equation (156) gives the perturbations in longitude equal to[14]

$$dv=\frac{1}{\sqrt{1-e^2}}\left\{\begin{aligned}&\frac{2d(rdr)}{a^2\,.ndt}+\frac{m'}{2}\cdot\Sigma\cdot\left\{D_i-D'_i\right\}e^2\,.\sin i\left(n't-nt+\epsilon'-\epsilon\right)\\ &+\frac{m'}{2}\cdot\Sigma\cdot E_i\,.ee'\,.\sin\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+\varpi-\varpi'\right\}\\ &-\frac{m'}{2}\cdot\Sigma\cdot E'_i\,.ee'\,.\sin\left\{i\left(n't-nt+\epsilon'-\epsilon\right)-\varpi-\varpi'\right\}\\ &-\frac{m'}{2}\cdot\Sigma\cdot\left\{\tfrac{1}{2}C_i+D_i\right\}.\,e^2\,.\sin\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+2nt+2\epsilon-2\varpi\right\}\\ &-\frac{m'}{2}\cdot\Sigma\cdot E_i\,.ee'\,.\sin\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+2nt+2\epsilon-\varpi-\varpi'\right\}\\ &-\frac{m'}{2}\Sigma\left\{\frac{(6-3i)n^2}{\left(i(n'-n)+2n\right)^2}\cdot aN+a^2\cdot\frac{dN}{da}\cdot\frac{2n}{i(n'-n)+2n}\right\}\times\\ &\qquad\sin\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+2nt+L\right\}\\ &-\frac{m'}{2}\Sigma\left\{\frac{2n}{i(n'-n)}\cdot a^2\cdot\frac{dN'}{da}-\frac{3n^2}{i(n'-n)^2}\cdot aN'\right\}\times\\ &\qquad\sin\left\{i\left(n't-nt+\epsilon'-\epsilon\right)+2nt+L'\right\}\end{aligned}\right\}. \qquad \textbf{(164)}$$

561. The inequalities of this order are very numerous, it is therefore necessary to select those that have the greatest values and to reject the rest, which can only be done in each particular case from the values of the divisors

$$i(n'-n)+3n,\ i(n'-n)+2n,\ i(n'-n)+n,\ i(n'-n).$$

For if the mean motions of the bodies m and m' be so nearly commensurable as to make any of these a small fraction, the inequality to which it is divisor will in general be of sufficient magnitude to be computed.

562. The inequalities in latitude will be determined afterwards.

Perturbations depending on the Cubes and Products of three Dimensions of the Eccentricities and Inclinations

563. These perturbations are only sensible when the divisor $i(n'-n)+3n,$ is a very small fraction, that is, when the mean motions of the two bodies are nearly commensurable; but as this divisor arises from the angle $i(n't-nt+\in'-\in)+3nt+3\in$ alone, the only part of the series R that is requisite by article 451, is[15]

$$\begin{aligned}
R = &+\frac{m'}{4}Q_0e'^3\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in-3v'\right\}\\
&+\frac{m'}{4}Q_1e'^2e\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in-2v'-v\right\}\\
&+\frac{m'}{4}Q_2e'e^2\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in-v'-2v\right\}\\
&+\frac{m'}{4}Q_3e^3\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in-3v\right\}\\
&+\frac{m'}{4}Q_4e'g^2\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in-v-2\Pi\right\}\\
&+\frac{m'}{4}Q_5eg^2\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in-v-2\Pi\right\}.
\end{aligned}$$

But

$$\begin{aligned}
&\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in-3v\right\}\\
&=+\cos 3v\ .\cos\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in\right\}\\
&\quad+\sin 3v\ .\sin\left\{i\left(n't-nt+\in'-\in\right)+3nt+3\in\right\};
\end{aligned}$$

each cosine may be resolved in the same manner; and if

$$P = \tfrac{1}{4}\left\{\begin{matrix} +Q_0 \,.\, e'^3 \,.\sin 3v' + Q_1 e'^2 e \sin(2v' + v) \\ +Q_2 e e'^2 \sin(v' + 2v) + Q_3 e^3 \sin 3v \\ +Q_4 e' g^2 \sin(2\Pi + v') + Q_5 e g^2 \sin(2\Pi + v) \end{matrix}\right\}, \qquad \textbf{(165)}$$

[and]

$$P' = \tfrac{1}{4}\left\{\begin{matrix} +Q_0 \,.\, e'^3 \,.\cos 3v' + Q_1 e'^2 e \cos(2v' + v) \\ +Q_2 e e'^2 \cos(v' + 2v) + Q_3 e^3 \cos 3v \\ +Q_4 e' g^2 \cos(2\Pi + v') + Q_5 e g^2 \cos(2\Pi + v) \end{matrix}\right\}. \qquad \textbf{(166)}$$

This part of R becomes

$$\begin{aligned} R = &+m'P\,.\sin\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon\} \\ &+ m'P'.\cos\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon\} \end{aligned} \qquad \textbf{(167)}$$

564. Let[16] $\dfrac{rdr}{a^2} = m'K\cos\{i(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon + B\}$ be the part of the equation (162) that has the divisor $i(n'-n)+3n$; by the substitution of this, and of the preceding value of R, equation (155) gives, when integrated,[17]

$$\begin{aligned} \frac{rdr}{a^2} = &-\frac{2(i-3)m'n}{i(n'-n)+3n}\left\{\begin{matrix} +aP\sin\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon\} \\ +aP'\cos\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon\} \end{matrix}\right\} \\ &-\tfrac{3}{2}m'eK\cos\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 2\epsilon + B - v\} \\ &+\tfrac{1}{2}m'eK\cos\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon + B + v\}; \end{aligned}$$

and because

$$\frac{rdr}{a^2} = \frac{r}{a}\cdot\frac{dr}{a} = \frac{r}{a}m'K\cos\{i(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon + B\}$$

the whole perturbations in the radius vector having the divisor $i(n'-n)+3n$, are

$$\begin{aligned} \frac{dr}{a} = &+m'K\cos\{i(n't - nt + \epsilon' - \epsilon) + 2nt + 2\epsilon + B\} \\ &-m'Ke\cos\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon - v + B\} \\ &+m'Ke\cos\{i(n't - nt + \epsilon' - \epsilon) + nt + \epsilon + v + B\} \\ &-\frac{2(i-3)nm'}{i(n'-n)+3n}\left\{\begin{matrix} +aP\,.\sin\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon\} \\ +aP'.\cos\{i(n't - nt + \epsilon' - \epsilon) + 3nt + 3\epsilon\} \end{matrix}\right\}. \end{aligned} \qquad \textbf{(168)}$$

565. If this quantity and the preceding value of R be substituted in equation (156) the result will be,

$$d\,v = -\frac{3(3-i)m'n^2}{\{i(n'-n)+3n\}^2}\begin{Bmatrix}+aP'.\sin\{i(n't-nt+\in'-\in)+3nt+3\in\}\\ -aP.\cos\{i(n't-nt+\in'-\in)+3nt+3\in\}\end{Bmatrix}$$
$$+\frac{2m'n}{i(n'-n)+3n}\begin{Bmatrix}+a^2\left(\frac{dP}{da}\right)\cos\{i(n't-nt+\in'-\in)+3nt+3\in\}\\ -a^2\left(\frac{dP'}{da}\right)\sin\{i(n't-nt+\in'-\in)+3nt+3\in\}\end{Bmatrix} \qquad \textbf{(169)}$$
$$-\frac{m'e}{2}K\sin\{i(n't-nt+\in'-\in)+3nt+3\in -\nu +B\}$$
$$+\frac{5}{2}m'eK\sin\{i(n't-nt+\in'-\in)+nt+\in +\nu +B\}.$$

And as that part of $d\,v$ article 560, having the divisor $i(n'-n)+3n$ is nearly[18]

$$dz = 2m'K\sin\{i(n't-nt+\in'-\in)+2nt+2\in +B\}$$

if [19] $2K = K'$ the term $\frac{5}{4}m'eK'.\sin\{i(n't-nt+\in'-\in)+nt+\in +\nu +B\}$ must replace the last term in the preceding value of $d\,v$, &c.

Secular Variation of the Elliptical Elements during the periods of the Inequalities

566. An inequality

$$\frac{C}{\{5n'-2n\}}\begin{matrix}\sin\\ \cos\end{matrix}\ \{(5n'-2n)t+B\}$$

is at its maximum when the sine or cosine is unity; and if $5n'-2n$ be a small fraction, the coefficient

$$\frac{C}{\{5n'-2n\}^2}$$

will be very great. The period of an inequality is the time the argument or angle $(5n'-2n)t+B$ takes to increase from zero to 360°; it is evident that the period will be the greater, the less the difference $5n'-2n$.

Thus, the perturbations in longitude expressed by

$$d\,v = -\frac{3(3-i)\,m'n^2}{\{i(n'-n)+3n\}^2}\cdot\begin{Bmatrix} +aP'\sin\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\} \\ -aP\cos\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\} \end{Bmatrix}$$

are very great, and of long periods, when $i(n'-n)+3n$ is a small fraction.

567. The square of the divisor could only be introduced by a double integration, consequently the preceding value of $d\,v$ is the integral of the part

$$d\,v = -3a\iint ndt\,.\,\mathrm{d}R$$

of equation (156), which is the periodic inequality in the mean motion of m, when troubled by m', in article 439. Thus, when the mean motions are nearly commensurable, all terms having the small divisors in question, must be applied as corrections to the mean motion of the troubled planet.

568. In some cases the periods of these inequalities extend to many centuries; in so long a time the secular variations of the elements of the orbits have a very sensible influence on these perturbations and in order to include this effect, the expression

$$d\,v = -3\iint andt\,.\,\mathrm{d}R$$

must be integrated *by parts* in the hypothesis of P and P' being variable functions of the elements. Now

$$R = +m'P\sin\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\}$$
$$+m'P'\cos\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\};$$

whence

$$\mathrm{d}R = +m'P\,.\,(3-i)\,ndt\,.\cos\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\}$$
$$-m'P'\,.\,(3-i)\,ndt\,.\sin\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\}$$

and

$$-3a\iint ndt\,.\,\mathrm{d}R = +3a(3-i)\,.\,m'\iint P'\,.\,n^2dt^2\,.\sin\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\}$$
$$-3a(3-i)\,.\,m'\iint P\,.\,n^2dt^2\,.\cos\{i(n't-nt+\epsilon'-\epsilon)+3nt+3\epsilon\}.$$

From the integration of this equation it will be found that the periodic inequality in the mean motion, depending on the third dimensions of the eccentricities and inclinations, and affected by the secular variations during its period, is

$$d\,v = dz \;= \frac{3(3-i)\,m'n^2}{\{i(n'-n)+3n\}^2}\times \left\{\begin{array}{l} +\left\{a'P - \dfrac{2a\,.\,dP'}{\{i(n'-n)+3n\}dt} - \dfrac{3a\,.\,d^2P}{\{i(n'-n)+3n\}^2\,dt^2} - \&c.\right\}\times \\ \quad \cos\{i(n't-nt+\in'-\in)+3nt+3\in\} \\ -\left\{a'P + \dfrac{2a\,.\,dP}{\{i(n'-n)+3n\}dt} - \dfrac{3a\,.\,d^2P'}{\{i(n'-n)+3n\}^2\,dt^2} + \&c.\right\}\times \\ \quad \sin\{i(n't-nt+\in'-\in)+3nt+3\in\} \end{array}\right\}. \qquad \textbf{(170)}$$

This correction must be applied to the mean motions in the elliptical part of such planets as have their motions nearly commensurable.

569. The same method of integration may be employed for the term in equation (164), that has the divisor $n^2 - \{i(n'-n)+2n\}^2$ when the quantity $i(n'-n)+3n$ is a small fraction, and in general to all inequalities of long periods having small divisors.

The variation of the elements during the periods of the inequalities may be estimated by the following approximate method, which will answer for several centuries before and after the epoch. By the method employed in article 563 the sum of the terms in equations (164) depending on the angle $i(n't-nt+\in'-\in)+2nt+2\in$ may be put under the form

$$d\,v = m'\overline{P}\sin\{i(n't-nt+\in'-\in)+2nt+2\in\} + m'\overline{P}'\cos\{i(n't-nt+\in'-\in)+2nt+2\in\};$$

$\overline{P}$ and $\overline{P}'$ being functions of the elements of the orbits of m and m' determined by observation for a given epoch, say 1750. Since $\overline{P}$ and $\overline{P}'$ are known quantities, let

$$\frac{\overline{P}'}{\overline{P}} = \tan\overline{E}, \text{ and } \sqrt{\overline{P}^2+\overline{P}'^2} = \overline{F}$$

$\sin\overline{E}$ having the same sign with $\overline{P}'$ and $\overline{P}'$ with $\overline{P}$; hence

$$d\,v = m'\overline{F}\sin\{i(n't-nt+\in'-\in)+2nt+2\in+\overline{E}\}$$

are the perturbations in question for the epoch 1750. Now if the time t be made equal to 500 in the expressions for the elements in article 480, values of P and P' will be found for the year 2250, with which new values of F and E may be computed for that era. Again, values of P and P' may be obtained from the same formulae for the year 2750, and by the method employed in article 480, the series

$$F = \bar{F} + \frac{d\bar{F}}{dt}t + \tfrac{1}{2}\frac{d^2\bar{F}}{dt^2}t^2 + \&c.$$
$$E = \bar{E} + \frac{d\bar{E}}{dt}t + \tfrac{1}{2}\frac{d^2\bar{E}}{dt^2}t^2 + \&c.$$

will give values of the variable coefficients for any time t during many centuries, consequently

$$dv = m'\left\{\bar{F} + \frac{d\bar{F}}{dt}t + \&c.\right\}\sin\left\{i\left(n't - nt + \epsilon' - \epsilon\right) + 2nt + 2\epsilon + \bar{E} + \frac{d\bar{E}}{dt}t^2 + \&c.\right\} \qquad \textbf{(171)}$$

will give the perturbations, including the secular variations in the elements of the orbits during their periods, $\bar{F}$, $\bar{E}$ and their differences being relative to the epoch 1750.

570. The formulae that have been obtained will give the places of all the planets at any instant with great accuracy, except those of Jupiter and Saturn, which are so remote from the rest, as to be almost beyond the sphere of their disturbing influence; but their proximity to one another, and their immense magnitude, render their mutual disturbances greater than those of any of the other planets. They may be regarded as forming with the sun a system by themselves; and as there are some circumstances in their motions peculiar to them alone, their theory will form a separate subject of consideration.

Notes

[1] The right hand side of the second equation *reads* $\frac{dR}{dx}$ in the 1st edition (published erratum).

[2] The second term *reads* $-\frac{d^2r}{rdt_2}$ in the 1st edition.

[3] The 1st edition expression *reads* $mh^2 = m\mathrm{E}^2 + \mathrm{E}h^2$.

[4] The 1st edition expression *reads* $pc^2 + c\mathrm{B}^2 + \mathrm{E}h^2 = mh^2$.

[5] The term $d\,.\left(mp\right)$ *reads*: "$.\,dmp$" in the 1st edition.

[6] The expression *reads* $\left(mh\right)^2 = r^2dv + dr^2$ in the 1st edition (published erratum).

[7] Punctuation added.

[8] Right hand parenthesis is omitted in the 1st edition.

[9] First term *reads* $-m'a(3g + a\left(\frac{dA_0}{da}\right).nt$ in the 1st edition.

[10] The second and third terms in the first equation *read* $2m'ag + \frac{1}{2}m'a^2\left(\frac{dA_0}{da}\right)$ in the 1st edition (published erratum).

[11] This *reads* "very very embarrassing" in the 1st edition (published erratum).

[12] See note 63, *Preliminary Dissertation.*

[13] See note 4, *Introduction.*

[14] A missing parenthesis in line 6 and missing accent on $\in$ in line 7 of the 1st edition *read* $i\left(n'-n\right)+2n)^2$ and $\sin\left\{i\left(n't-nt+\in-\in\right)+2nt+L'\right\}$.

[15] Punctuation added.

[16] This *reads* $\frac{rd\,r}{a^2}=m'K\cos\left\{i\left(n't-n't+\in-\in\right)+2nt+2\in+B\right\}$ in the 1st edition.

[17] Right hand parenthesis is omitted in the 1st edition.

[18] This *reads* $m'He$ *for* $2m'K$ in the 1st edition (published erratum).

[19] In the 1st edition, the remaining two lines in this article *read*:

"the terms $\frac{5}{4}m'eK\,.\sin\left\{i\left(n't-nt+\in'-\in\right)+nt+\in+v+B\right\}$ must be added to the preceding value of $d\,v$, which will then be the whole perturbations in longitude having the divisor in question." (published erratum).