BOOK II

CHAPTER XI

INEQUALITIES OCCASIONED BY THE ELLIPTICITY OF THE SUN

592. As the sun has hitherto been considered a sphere, his action was assumed to be the same as if his mass were united in his centre of gravity; but from his rotatory motion, his form must be spheroidal on account of his centrifugal force, therefore the excess of matter at his equator may have an influence on the motions of the planets.

In the theory of spheroids it is found that the attraction of the redundant matter at the equator is expressed by

$$\left(r-\tfrac{1}{2}y\right)\cdot\frac{R'^2}{r^3}\left(h^2-\tfrac{1}{3}\right).$$

Where r is the ellipticity of the sun, y the ratio of the centrifugal force to gravity at the solar equator, h the declination of a planet m relative to this equator, R' the semidiameter of the sun, his mass being unity. Therefore, the attraction of the elliptical part of the sun's mass adds the term

$$\left(r-\tfrac{1}{2}y\right)\cdot\frac{R'^2}{r^3}\left(h^2-\tfrac{1}{3}\right)$$

to the disturbing action expressed by the series R in article 449. If this disturbing action of the sun's spheroidal form be alone considered, omitting h^2, and substituting

$$\frac{1}{a^3}\left(1-\tfrac{3}{2}e^2\right),\ \text{for } r^{-3},$$

it gives, with regard to secular quantities alone,

$$F=-\tfrac{1}{3}\left(r-\tfrac{1}{2}y\right).\frac{R'^2}{a^3}\left(1-\tfrac{3}{2}e^2\right),$$

and

$$\frac{dF}{de}=e\left(r-\tfrac{1}{2}y\right)\cdot\frac{R'^2}{a^3}.$$

The substitution of which in

$$dv=\frac{andt}{e}\cdot\frac{dF}{de},$$

gives by integration,

$$dv = \left(r - \tfrac{1}{2}y\right)\cdot\frac{R'^2}{a^2}\cdot nt\,.$$

Thus the action of the excess of matter at the sun's equator produces a direct motion in the perihelia of the planetary orbits.

593. The effect of the sun's ellipticity on the position of the orbits may be ascertained from the last of equations (115), or

$$dp = andt\cdot\frac{dF}{dq}\,.$$

Since h is the declination of the planet m on the plane of the sun's equator, if the equator be taken as the fixed plane, then will

$$h^2 = \frac{z^2}{r^2}\,.$$

And if the eccentricity be omitted,

$$F = \left(r - \tfrac{1}{2}y\right)\cdot\frac{R'^2}{a^5}\left(z^2 - a^2\right),$$

therefore

$$\frac{dF}{dz} = 2\,.\left(r - \tfrac{1}{2}y\right)\cdot\frac{R'^2}{a^5}\cdot z\,.$$

But

$$\frac{dF}{dq} = \frac{dF}{dz}\cdot\frac{dz}{dq} = \frac{dF}{dz}\cdot a\sin\left(nt + \in\right)$$

On account of equation,

$$\frac{z}{a} = q\,.\sin\left(nt + \in\right) - p\cos\left(nt + \in\right)$$

consequently,

$$\frac{dF}{dq} = 2.\left(r - \tfrac{1}{2}y\right)\cdot\frac{R'^2}{a^5}\cdot z\,.\sin\left(nt + \in\right)$$

or substituting $a\,.\tan f\,.\sin\left(nt + \in - q\right)$ for *z*,

$$\frac{dF}{dq} = -\left(r - \tfrac{1}{2}y\right)\cdot\frac{R'^2}{a^3}\cdot\cos q\,.\tan f,$$

whence

$$dp = -ndt\,.\left(r - \tfrac{1}{2}y\right)\cdot\frac{R'^2}{a^2}\cdot\cos q\,.\tan f\,.$$

But

$$p = \tan f \, . \sin q \;;$$

whence

$$dp = dq \, . \tan f \, . \cos q \; .$$

Therefore

$$dq = -ndt \, . \left(r - \tfrac{1}{2} y \right) \cdot \frac{R'^2}{a^2} ,$$

and

$$dq = -nt \, . \left(r - \tfrac{1}{2} y \right) \cdot \frac{R'^2}{a^2} .$$

Thus the nodes of the planetary orbits have a retrograde motion on the plane of the solar equator equal to the direct motion of their perihelia on the same plane, both so small that they are scarcely perceptible even in Mercury. As neither the eccentricities nor the inclinations are affected by this disturbance, it has no influence on the stability of the system.