BOOK III

CHAPTER II

NUMERICAL VALUES OF THE COEFFICIENTS

735. THE following data are obtained by observation.

$$m = 0.0748013$$
$$e = 0.05486281$$
$$g = 0.0900807$$
$$c = 0.99154801$$
$$g = 1.00402175$$
$$e' = 0.016814, \text{ at the epoch 1750,}$$
$$m = \frac{1}{75}.$$

e and g result from the comparison of the coefficients of the sines of the angles $cv - v$ and $gv - q$, computed from observation with those from theory. With these data equation (230) gives

$$\frac{1}{a} = \frac{1}{a_1} \cdot 0.9973020; \quad \frac{a^2}{\sqrt{a_1}} = 1.0003084 = \frac{1}{n};$$

whence

$$\frac{1}{a} = \sqrt[3]{\frac{n^2 \left(1.0003084\right)^2}{0.9973020}} .$$

With these the formulae of articles 718 and 726 and 734 give

$A_0 = +0.0070962$	$A_{11} = +0.349187$
$A_1 = +0.201816$	$A_{12} = +0.0026507$
$A_2 = -0.00372953$	$A_{13} = +0.0077734$
$A_3 = -0.00300427$	$A_{14} = -0.012989$
$A_4 = +0.0284957$	$A_{15} = -0.742373$
$A_5 = -0.00591628$	$A_{16} = -0.041378$
$A_6 = -0.0698493$	$A_{17} = -0.113197$
$A_7 = +0.516751$	$A_{18} = +1.08469$
$A_8 = -0.20751$	$A_{19} = +0.001601$

$A_9 = +0.274122$	$B_0 = +0.0282636$
$A_{10} = +0.0008107$	$B_1 = -0.0000024$
$B_2 = -0.0055075$	$C_4 = +0.722823$
$B_3 = +0.019553$	$C_5 = -0.250034$
$B_4 = +0.0063661$	$C_6 = -0.0091988$
$B_5 = -0.0013668$	$C_7 = -0.414046$
$B_6 = -0.021272$	$C_8 = +0.0129865$
$B_7 = +0.07824$	$C_9 = +0.0039255$
$B_8 = -0.0833684$	$C_{10} = -0.0387853$
$B_9 = -0.0327678$	$C_{11} = +0.196755$
$B_{10} = +0.0720448$	$C_{12} = +0.12765$
$B_{11} = +0.491954$	$C_{13} = -1.081734$
$B_{12} = +0.0061023$	$C_{14} = +0.373115$
$B_{13} = +0.0920621$	$C_{15} = -0.616738$
$B_{14} = -0.0125619$	$C_{16} = +0.272377$
$B_{15} = +0.0038663$	$C_{17} = +0.033825$
$C_0 = -2.003974$	$C_{18} = +0.173647$
$C_1 = +0.752886$	$C_{19} = -0.236616$
$C_2 = -0.336175$	$C_{20} = -2.16938$
$C_3 = +0.243118$	

736. If these coefficients be reduced to sexagesimal seconds, the mean longitude of the moon will become[1]

$$
\begin{aligned}
nt + \epsilon = {} & v + \tfrac{3}{2} m^2 \int \left(e'^2 - \overline{e}'^2 \right) dv \\
& - 22{,}677''.5 \cdot \sin\left(cv - v\right) \\
& + \quad 467''.42 \cdot \sin\left(2cv - 2v\right) \\
& - \quad 11''.45 \cdot \sin\left(3cv - 3v\right) \\
& + \quad 406''.92 \cdot \sin\left(2gv - 2q\right) \\
& + \quad 66''.37 \cdot \sin\left(2gv - cv - 2q + v\right) \\
& - \quad 22''.96 \cdot \sin\left(2gv + cv - 2q - v\right) \\
& - \quad 1{,}906''.93 \cdot \sin\left(2v - 2mv\right) \\
& - \quad 4{,}685''.46 \cdot \sin\left(2v - 2mv - cv + v\right) \\
& + \quad 147''.68 \cdot \sin\left(2v - 2mv + cv - v\right)
\end{aligned}
\qquad \textbf{(240)}
$$

$$
\begin{aligned}
&+ \quad 13''.61 \cdot \sin\left(2v-2mv+c'mv-v'\right)\\
&- \quad 134''.51 \cdot \sin\left(2v-2mv-c'mv+v'\right)\\
&+ \quad 682''.37 \cdot \sin\left(c'mv-v'\right)\\
&+ \quad 24.29 \cdot \sin\left(2v-2mv-cv+c'mv+v-v'\right)\\
&- \quad 205.82 \cdot \sin\left(2v-2mv-cv-c'mv+v+v'\right)\\
&+ \quad 70.99 \cdot \sin\left(cv+c'mv-v-v'\right)\\
&- \quad 117.35 \cdot \sin\left(cv-c'mv-v+v'\right)\\
&+ \quad 169.09 \cdot \sin\left(2cv-2v+2mv-2v\right)\\
&+ \quad 56.62 \cdot \sin\left(2gv-2v+2mv-2q\right)\\
&+ \quad 10.13 \cdot \sin\left(2c'mv-2v'\right)\\
&+ \quad 122.014 \cdot (1+i)\sin\left(v-mv\right)\\
&- \quad 18.81 \cdot (1+i)\sin\left(v-mv+c'mv-v'\right).
\end{aligned}
$$

737. The two last terms have been determined in supposing

$$\frac{a}{a'} = \frac{(1+i)}{400}.$$

This fraction is the ratio of the parallax of the sun to that of the moon; it differs very little from $\frac{1}{400}$, but for greater generality it is multiplied by the indeterminate coefficient $1+i$; and by comparing the coefficient of $\sin(v-mv)$ with the result of observations the solar parallax is obtained, as will be shown afterwards.

738. It has been shown that the action of the moon produces the inequality

$$m \cdot \frac{a}{a'} \sin(v-mv)$$

in the earth's longitude. This action of the moon changes the earth's place, and, consequently, the moon's place with regard to the sun, so that the moon indirectly troubles her own motion, producing in her mean longitude the inequality

$$0.54139 \cdot m \cdot \frac{a}{a'} \sin(v-nv).$$

Thus the direct action of the moon is weakened by reflection in the ratio of 0.54139 to unity.

739. Equation (233) gives the tangent of the latitude, but the expression of the arc by the tangent s is

$$s-\tfrac{1}{3}s^3+\tfrac{1}{5}s^5-\&c.$$

Thus the latitude is nearly

$$g\left(1-\tfrac{1}{4}g^2\right)\sin\left(gv-q\right)+d\,s\times$$
$$\left\{1-\tfrac{1}{2}g^2+\tfrac{1}{2}g^2\cos\left(2gv-2q\right)+\tfrac{1}{12}g^3\sin\left(3gv-3q\right)\right\}.$$

And from the preceding data the latitude of the moon is easily found to be

$$\begin{aligned}
s = &+18{,}542''.0\,.\sin\left(gv-q\right) \\
&+\quad 12''.57\,.\sin\left(3gv-3q\right) \\
&+\quad 525''.23\,.\sin\left(2v-2mv-gv+q\right) \\
&+\quad 1''.14\,.\sin\left(2v-2mv+gv-q\right) \\
&-\quad 5''.53\,.\sin\left(gv+cv-q-v\right) \\
&+\quad 19''.85\,.\sin\left(gv-cv-q+v\right) \\
&+\quad 6''.46\,.\sin\left(2v-2mv-gv+cv+q-v\right) \\
&-\quad 1''.39\,.\sin\left(2v-2mv+gv-cv-q+v\right) \\
&-\quad 21''.6\,.\sin\left(2v-2mv-gv-cv+q+v\right) \\
&+\quad 24''.34\,.\sin\left(gv+c'mv-q-v'\right) \\
&-\quad 25''.94\,.\sin\left(gv-c'mv-q+v'\right) \\
&-\quad 10''.2\,.\sin\left(2v-2mv-gv+c'mv+q-v'\right) \\
&+\quad 22''.42\,.\sin\left(2v-2mv-gv-cmv+q+v'\right) \\
&+\quad 27''.41\,.\sin\left(2cv-gv-2v+q\right) \\
&+\quad 5''.29\,.\sin\left(2cv+gv-2v+2mv-2v-q\right).
\end{aligned} \qquad \textbf{(241)}$$

740. The sine of the horizontal parallax of the moon is

$$\frac{R^{`}}{r}=\frac{R^{`}u}{\sqrt{1+ss}},$$

$R^{`}$ being the terrestrial radius, but as this arc is extremely small, it may be taken for its sine; hence, if

$$\frac{1}{a}\left\{1+e^2+\tfrac{1}{4}g^2+e\left(1+e^2\right)\cos\left(cv-v\right)-\tfrac{1}{4}g^2\cos\left(2gv-2q\right)\right\}+du$$

be put for m, and quantities of the order $\frac{R`}{a}e^4$ rejected, the parallax will be

$$\frac{R`}{r}=\frac{R`u}{\sqrt{1+s^2}}=\frac{R`}{a}\left(1+e^2\right)\left\{1+e\left[1-\tfrac{1}{4}g^2+\tfrac{1}{4}g^2\cos\left(2gv-2q\right)\right]\cos\left(cv-v\right)+adu-sds\right\}.$$

In the untroubled orbit of the moon the radius vector, and, consequently, the parallax, varies according to a fixed law through every point of the ellipse. Its mean value, or the constant part of the horizontal parallax, is $\frac{R`}{a}$, to which the rest of the series is applied as corrections arising both from the ellipticity of the orbit and the periodic inequalities to which it is subject.

741. In order to compute the constant part of the parallax, let s be the space described by falling bodies in a second in the latitude, the square of whose sine is $\tfrac{1}{3}$, l and $R`$ the corresponding lengths of the pendulum and terrestrial radius, p the ratio of the semicircumference to the radius, E and m the masses of the earth and moon; then, supposing

$$E+m=1,$$

$$\frac{E}{\left(E+m\right)R`^2}=2s=p^2l,\text{ also } n=\frac{2p}{T},$$

T being the number of seconds a sidereal revolution of the moon; and by article 735

$$\frac{1}{a}=\sqrt[3]{\frac{n^2\left(1.0003084\right)^2}{0.9973020}},$$

therefore

$$\frac{R`}{a}=\sqrt[3]{\frac{E}{E+m}\cdot\frac{R`}{l}\cdot\frac{4\left(1.0003084\right)^2}{T^2 0.997320}}.$$

Now the length of the pendulum, independent of the centrifugal force, is

$$l=32.648\text{ feet},$$

also

$$R`=20{,}898{,}500\text{ feet},$$
$$T=2{,}360{,}591''.8;$$

and if

$$m=\frac{E}{58.6}$$

it will be found that

$$\frac{R\text{`}}{a} = 0.01655101\text{, and therefore } \frac{R\text{`}}{a}\left(1+e^2\right) = 3{,}424''.16\,;$$

this value augmented by $3''.74$, to reduce it to the equator, is $3{,}427''.9$; hence the equatorial parallax of the moon in functions of its true longitude is

$$
\begin{aligned}
\frac{1}{r} = {} & +3{,}427''.9 \\
& + \; 187''.48\cos\left(cv - v\right) \\
& + \; 24''.68\cos\left(2v - 2mv\right) \\
& + \; 47''.92\cos\left(2v - 2mv - cv + v\right) \\
& - \; 0''.7\;\cos\left(2v - 2mv + cv - v\right) \\
& - \; 0''.17\cos\left(2v - 2mv + c'mv - v'\right) \\
& + \; 1''.64\cos\left(2v - 2mv - c'mv + v'\right) \\
& - \; 0''.33\cos\left(c'mv - v'\right) \\
& - \; 0''.22\cos\left(2v - 2mv - cv + c'mv + v - v'\right) \\
& + \; 1''.63\cos\left(2v - 2mv - cv - cmv + v + v'\right) \\
& - \; 0''.45\cos\left(cv + c'mv - v - v'\right) \\
& + \; 0''.86\cos\left(cv - c'mv - v + v'\right) \\
& + \; 0''.01\cos\left(2cv - 2v\right) \\
& + \; 3''.6\cos\left(2cv - 2v + 2mv - 2v\right) \\
& + \; 0''.07\cos\left(2gv - 2q\right) \\
& - \; 0''.18\cos\left(2gv - 2v + 2mv - 2q\right) \\
& - \; 0''.01\cos\left(2c'mv - 2v'\right) \\
& - \; 0''.95\cos\left(2gv - cv - 2q + v\right) \\
& - \; 0''.06\cos\left(2v - 2mv - 2gv + cv + 2q - v\right) \\
& - \; 0''.97\left(1+i\right)\cos\left(v - mv\right) \\
& + \; 0''.16\left(1+i\right)\cos\left(v - mv + c'mv - v'\right) \\
& - \; 0''.04\cos\left(2v - 2mv + 2cv - 2c'mv - v + v'\right) \\
& - \; 0''.15\cos\left(4v - 4mv - cv + v\right) \\
& + \; 0''.05\cos\left(4v - 4mv - cv + 2v\right) \\
& + \; 0''.13\cos\left(2cv - 2v + 2mv + c'mv - 2v - v'\right)
\end{aligned}
\qquad \textbf{(242)}
$$

$$
\begin{aligned}
+ &\quad 0''.02\cos\left(2cv+2v-2mv-2\varpi\right) \\
- &\quad 0''.12\left(1+i\right)\cos\left(cv-v+mv-\varpi\right)
\end{aligned}
$$

The greatest value of the parallax is $1^\circ 1' 29''.32$, which happens when the moon is in perigee and opposition; the least, $58' 29''.93$, happens when the moon is in apogee and conjunction.

742. With $m = \dfrac{E}{74}$, Mr. Damoiseau[2] finds the constant part of the equatorial parallax equal to $3{,}431''.73$.

743. The lunar parallax being known, that of the sun may be determined by comparing the coefficients of the inequality

$$122''.014\left(1+i\right)\sin\left(v-mv\right)$$

in the moon's mean longitude with the same derived from observation. In the tables of Burg,[3] reduced from the true to the mean longitude, this coefficient is $122''.378$; hence

$$i+1=\frac{122''.378}{122''.014}=1''.00298, \text{ and } \frac{a}{a_{/}}=\frac{1''.00298}{400}.$$

But the solar parallax is

$$\frac{R`}{a'}=\frac{R`}{a}\frac{a}{a'}=\frac{R`}{a}\cdot\frac{1''.00298}{400},$$

but

$$\frac{R`}{a}=0.01655101,$$

hence

$$\frac{R`}{a'}=\frac{1''.00298\times 0.01655101}{400}=8''.5602,$$

which is the mean parallax of the sun in the parallel of latitude, the square of whose sine is $\frac{1}{3}$.

Burckhardt's tables[4] give $122''.97$ for the value of the coefficient, whence the solar parallax is $8''.637$, differing very little from the value deduced from the transit of Venus. This remarkable coincidence proves that the action of the sun upon the moon is very nearly equal to his action on the earth, not differing more than the three millionth part.

744. The constant part of the lunar parallax is $3{,}432''.04$, by the observations of Mr. Maskelyne,[5] consequently the equation

$$3{,}432''.04 = \sqrt[3]{\frac{E}{E+m} \cdot \frac{R`}{l} \cdot \frac{4(1.0003084)^2}{T^2(0.9973020)}}$$

gives the mass of the moon equal to

$$\frac{1}{74.2}$$

of that of the earth.

Since by article 646, $\frac{R`}{a} = 0.01655101$, in the latitude the square of whose sine is $\frac{1}{3}$; if $R`$, the mean radius of the earth, be assumed as unity, the mean distance of the moon from the earth is 60.4193 terrestrial radii, or about 247,583 English miles.

745. As theory combined with observations with the pendulum, and the mensuration of the degrees of the meridian, give a value of the lunar parallax nearly corresponding with that derived from astronomical observations, we may reciprocally determine the magnitude of the earth from these observations; for if the radius of the earth be assumed as the unknown quantity in the expression in article 646, it will give its value equal to 20,897,500 English feet.

'*Thus,*' says Laplace,[6] '*an astronomer, without going out of his observatory, can now determine with precision the magnitude and distance of the earth from the sun and moon, by a comparison of observations with analysis alone; which in former times it required long voyages in both hemispheres to accomplish.*'

746. The apparent diameter of the moon varies with its parallax, for if P be the horizontal parallax, $R`$ the terrestrial radius, r the radius vector of the moon, D her real, and A her apparent diameters; then

$$P = \frac{R`}{r},\ A = \frac{D}{r};\ \text{whence}\ \frac{P}{A} = \frac{R`}{D}$$

a ratio that is constant if the earth be a sphere. It is also constant at the same point of the earth's surface, whatever the figure of the earth may be.

$$\text{If}\ P = 56'4''.168\ \text{and}\ \tfrac{1}{2}A = 31'7''.73;$$

then

$$\frac{A}{2P} = 0.27293 = \tfrac{3}{11}\ \text{nearly};$$

thus if $\frac{11}{3}$ be multiplied by the moon's apparent semidiameter, the corresponding horizontal parallax will be obtained.

Secular Inequalities in the Moon's Motions

747. It has been shown, that the action of the planets is the cause of a secular variation in the eccentricity of the earth's orbit, which variation produces analogous inequalities in the mean motion of the moon, in the motion of her perigee and in that of her nodes.

The Acceleration

748. The secular variation in the mean motion of the moon denominated the Acceleration, was discovered by Halley;[7] but Laplace[8] first showed that it was occasioned by the variation in the eccentricity in the earth's orbit. The acceleration in the mean motion of the moon is ascertained by comparing ancient with modern observations; for if the ancient observations be assumed as observed longitudes of the moon, a calculation of her place for the same epoch from the lunar tables will render the acceleration manifest, since these tables may be regarded as data derived from modern observations.

An eclipse of the moon observed by the Chaldeans at Babylon, on the 19th of March, 721 years before the Christian era, which began about an hour after the rising of the moon, as recorded by Ptolemy,[9] has been employed. As an eclipse can only happen when the moon is in opposition, the instant of opposition may be computed from the solar tables, which will give the true longitude of the moon at the time, and the mean longitude may be ascertained from the tables. Now, if we compare this result with another mean longitude of the moon computed from modern observations, the difference of the longitudes augmented by the requisite number of circumferences will give the arc described by the moon parallel to the ecliptic during the interval between the observations, and the mean motion of the moon during 100 Julian years may be ascertained by dividing this arc by the number of centuries elapsed. But the mean motion thus computed by Delambre,[10] Bouvard,[11] and Burg, is more than 200″ less than that which is derived from a comparison of modern observations with one another. The same results are obtained from two eclipses observed by the Chaldeans in the years 719 and 720 before the Christian era. This acceleration was confirmed by comparing less ancient eclipses with those that happened recently; for the epoch of intermediate observations being nearer modern times, the differences of the mean longitudes ought to be less than in the first case, which is perfectly confirmed, by the eclipses observed by Ibn-Junis,[12] an Arabian astronomer of the eleventh century. It is therefore proved beyond a doubt, that the mean motion of the moon is accelerated, and her periodic time consequently diminished from the time of the Chaldeans.

Were the eccentricity of the terrestrial orbit constant, the term

$$\tfrac{3}{2}m^2\int\left(e'^2-\overline{e}^2\right)dv$$

would be united with the mean angular velocity of the moon; but the variation of the eccentricity, though small, has in the course of time a very great influence on the lunar motions. The mean motion of the moon is accelerated, when the eccentricity of the earth's orbit diminishes, which it has continued to do from the most ancient observations down to our times; and it will continue to be accelerated until the eccentricity begins to increase, when it will be retarded. In the interval between 1750 and 1850, the square of the eccentricity of the terrestrial orbit has diminished by 0.00000140595. The corresponding increment in the angular velocity of the moon is the

0.000000011782th part of this velocity. As this increment takes place gradually and proportionally to the time, its effect on the motion of the moon is less by one half than if it had been uniformly the same in the whole course of the century as at the end of it. In order, therefore, to determine the secular equation of the moon at the end of a century estimated from 1801, we must multiply the secular motion of the moon by half the very small increment of the angular velocity; but in a century the motion of the moon is $1{,}732{,}559{,}351''.514$, which gives $10''.2065508$ for her secular equation. Assuming that for 2000 years before and after the epoch 1750, the square of the eccentricity of the earth's orbit diminishes as the time, the secular equation of the mean motion will increase as the square of the time: it is sufficient then during that period to multiply $10''.2065508$ by the square of the number of centuries elapsed between the time for which we compute and the beginning of the nineteenth century; but in computing back to the time of the Chaldeans, it is necessary to carry the approximation to the cube of the time. The numerical formula for the acceleration is easily found, for since[13]

$$\tfrac{3}{2}m^2\int\left(e'^2-\bar{e}^2\right)dv$$

is the acceleration in the mean longitude of the moon, the true longitude of the moon in functions of her mean longitude will contain the term

$$-\tfrac{3}{2}m^2\int\left(e'^2-\bar{e}^2\right)ndt,$$

$\bar{e}$ being the eccentricity of the terrestrial orbit at the epoch 1750. If then, t be any number of Julian years from 1750, by article 480,

$$2e'=2\bar{e}-0''.171793t-0''.000068194t^2$$

is the eccentricity of the earth's orbit at any time t, whence the acceleration is

$$10''.1816213\,.\,T^2+0''.018538444\,.\,T^3,$$

T being any number of centuries before or after 1801.

In consequence of the acceleration, the mean motion of the moon is $7'\,30''$ greater in a century now than it was 2,548 years ago.

Motion of the Moon's Perigee

749. In the first determination of the motion of the lunar perigee, the approximation had not been carried far enough, by which the motion deduced from theory was only one half of that obtained by observation; this led Clairaut[14] to suppose that the law of gravitation was more complicated than the inverse ratio of the squares of the distance; but Buffon[15] opposed him on the principle that, the primordial laws of nature being the most simple, could only depend on one principle, and therefore their expression could only consist of one term. Although such reasoning is not always conclusive, Buffon was right in this instance, for, upon carrying the approximation to the squares of the disturbing force, the law of gravitation gives the motion of the lunar perigee exactly conformable to observation, for $\overline{e}'^2$ being the eccentricity of the terrestrial orbit at the epoch, the equation $c = \sqrt{1 - p - p'\overline{e}^2}$ when reduced to numbers is $c = 0.991567$, consequently $(1-c)v$ the motion of the lunar perigee is $0.008433.v$; and with the value of c in article 735 given by observation, it is $0.008452.v$, which only differs from the preceding by 0.000019. In Damoiseau's theory[16] it is $0.008453.v$, which does not differ much from that of Laplace. The terms depending on the squares of the disturbing force have a very great influence on the secular variation in the motion of the lunar perigee; they make its value three times as great as that of the acceleration: for the secular inequality in the lunar perigee is

$$\frac{p'}{2\sqrt{1+p}}\int\left(e'^2 - \overline{e}^2\right)ndt,$$

or, when the coefficient is computed, it is

$$3.00052\tfrac{3}{2}m^2\int\left(e'^2 - \overline{e}^2\right)ndt,$$

and has a contrary sign to the secular equation in the mean motion.

The motion of the perigee becomes slower from century to century, and is now $8'.2$ slower than in the time of Hipparchus.[17]

Motion of the Nodes of the Lunar Orbit

750. The sidereal motion of the node on the true ecliptic as determined by theory, does not differ from that given by observation by a 350th part; for the expression in article 727 gives the retrograde motion of the node equal to $0.00400105v$, and by observation

$$(g-1)v = 0.00402175v,$$

the difference being 0.00001125. Mr. Damoiseau makes it

$$g - 1 = 0.0040215.$$

The secular inequality in the motion of the node depends on the variation in the eccentricity of the terrestrial orbit, and has a contrary sign to the acceleration. Its analytical expression gives

$$\frac{q'}{2\sqrt{1+q}}\int\left(e'^2 - \overline{e}^2\right)dv = 0.735452\tfrac{3}{2}m^3\int\left(e'^2 - \overline{e}^2\right)dv.$$

As the motion of the nodes is retrograde, this inequality tends to augment their longitudes posterior to the epoch.

751. It appears from the signs of these three secular inequalities, as well as from observation, that the motion of the perigee and nodes become slower, whilst that of the moon is accelerated; and that their inequalities are always in the ratio of the numbers 0.735452, 3.00052, and 1.

752. The mean longitude of the moon estimated from the first point of Aries is only affected by its own secular inequality; but the mean anomaly estimated from the perigee, is affected both by the secular variation of the mean longitude, and by that of the perigee; it is therefore subject to the secular inequality $-4.00052\frac{3}{2}m^2\int\left(e'^2-\bar{e}^2\right)dv$ more than four times that of the mean longitude. From the preceding values it is evident that the secular motion of the moon with regard to the sun, her nodes, and her perigee, are as the numbers 1; 0.265; and 4; nearly.

753. At some future time, these inequalities will produce variations equal to a fortieth part of the circumference in the secular motion of the moon; and in the motion of the perigee, they will amount to no less than a thirteenth part of the circumference. They will not always increase: depending on the variation of the eccentricity of the terrestrial orbit they are periodic, but they will not run through their periods for millions of years. In process of time, they will alter all those periods which depend on the position of the moon with regard to the sun, to her perigee, and nodes; hence the tropical, synodic, and sidereal revolutions of the moon will differ in different centuries, which renders it vain to attempt to attain correct values of them for any length of time.

Imperfect as the early observations of the moon may be, they serve to confirm the results that have been detailed, which is surprising, when it is considered that the variation of the eccentricity of the earth's orbit is still in some degree uncertain, because the values of the masses of Venus and Mars are not ascertained with precision; and it is worthy of remark, that in process of time the development of the secular inequalities of the moon will furnish the most accurate data for the determination of the masses of these two planets.

754. The diminution of the eccentricity of the earth's orbit has a greater effect on the moon's motions than on those of the earth. This diminution, which has not altered the equation of the centre of the sun by more than $8'.1$ from the time of the most ancient eclipse on record, has produced a variation of $1^\circ\,8'$ in the longitude of the moon, and of $7^\circ.2$ in her mean anomaly.

Thus the action of the sun, by transmitting to the moon the inequalities produced by the planets on the earth's orbit, renders this indirect action of the planets on the moon more considerable than their direct action.

755. The mean action of the sun on the moon contains the inclination of the lunar orbit on the plane of the ecliptic; and as the position of the ecliptic is subject to a secular variation, from the action of the planets, it might be expected to produce a secular variation in the inclination of the moon's orbit. This, however, is not the case, for the action of the sun retains the lunar orbit at the same inclination on the orbit of the earth; and thus in the secular motion of the ecliptic, the orbit of the earth carries the orbit of the moon along with it, as it will be demonstrated, the change in the ecliptic affecting only the declination of the moon. No perceptible change has been observed in the inclination of the lunar orbit since the time of Ptolemy, which confirms the result of theory.

756. Although the inclination of the orbit does not vary from the change in the plane of the ecliptic; yet, as the expressions which determine the inclination and eccentricity of the lunar orbit, the parallax of the moon, and generally the coefficients of all the moon's inequalities, contain the eccentricity of the terrestrial orbit, they are all subject to secular inequalities corresponding to the secular variation of that quantity. Hitherto they have been insensible, but in the course of time will increase to an estimable quantity. Even now, it is necessary to include the effects of this variation in the inequality called the annual equation, when computing ancient eclipses.

757. The three co-ordinates of the moon have been determined in functions of the true longitudes, because the series converge better, but these quantities may be found in functions of the mean longitudes by reversion of series. For if nt, v, q, and $\in$, represent the mean motion of the moon, the longitudes of her perigee, ascending node and epoch, at the origin of the time, together with their secular equations for any time t, equation (240) becomes[18]

$$v-(nt+\in)=-\left\{C_0\,.\,e\,.\sin(cv-v)+C_1\,.\,e^2\sin 2(cv-v)+C_2\,.\,e^3\,.\sin 3(cv-v)+\&\text{c.}\right\}$$

or to abridge

$$v-(nt+\in)=S\,.$$

The general term of the series is

$$Q\cdot\sin(xv+y)\,.$$

And if Q' be the sum of the coefficients arising from the square of the series S, and depending on the angle $xv+y'$; Q'' the sum of the coefficients arising from the cube of S, and depending on the angle $xv+y$, &c. &c., the general term of the new series, which gives the true longitude of the moon in functions of her mean longitude, is

$$-\left\{Q+\tfrac{1}{2}x\,.\,Q'-\tfrac{1}{6}x^2\,.\,Q''-\tfrac{1}{24}x^3\,.\,Q'''+\&\text{c.}\right\}.\sin\left(x(nt+\in)+y\right)$$

Laplace does not give this transformation, but Damoiseau[19] has computed the coefficients for the epoch of January 1st, 1801, and has found that the true longitude of the moon in functions of its mean longitude $nt+\in=l$ is

$$\begin{aligned} v=nt+\in &+22{,}639''.7\,.\sin\{cl-v\}\\ &+\quad 768''.72\,.\sin(2cl-2v)\\ &+\quad 36''.94\,.\sin(3cl-3v)\\ &-\quad 411''.67\,.\sin(2gl-2q) \end{aligned}$$

$$
\begin{aligned}
&+ \quad 39''.51 . \sin(cl - 2gl - v + 2q) \\
&- \quad 45''.12 . \sin(cl + 2gl - v - 2q) \\
&+ \quad 2{,}370''.00 . \sin(2l - 2ml) \\
&+ \quad 4{,}589''.61 . \sin(2l - 2ml - cl + v) \\
&+ \quad 192''.22 . \sin(2l - 2ml + cl - v) \\
&- \quad 24''.82 . \sin(2l - 2ml + c'ml - v') \\
&+ \quad 165''.56 . \sin(2l - 2ml - c'ml + v') \\
&- \quad 673''.70 . \sin(c'ml - v') \\
&- \quad 28''.67 . \sin(2l - 2ml - cl + c'ml + v - v') \\
&+ \quad 207''.09 . \sin(2l - 2ml - cl - c'ml + v + v') \\
&- \quad 109''.27 . \sin(cl + cml - v - v') \\
&+ \quad 147''.74 . \sin(cl - cml - v + v') \\
&+ \quad 211''.57 . \sin(2l - 2ml - 2cl + 2v) \\
&+ \quad 54''.83 . \sin(2l - 2ml - 2gl + 2q) \\
&- \quad 7''.34 . \sin(2c'ml - 2v') \\
&- \quad 122''.48 . \sin(l - ml) \\
&- \quad 17''.56 . \sin(l - ml + c'ml - v').
\end{aligned}
$$

This is only the transformation of Laplace's equation (240), but Damoiseau[20] carries the approximation much farther.

758. The first term of this series is the mean longitude of the moon, including its secular variation.

The second term[21]

$$22{,}639''.7 \sin(cl - v)$$

is the equation of the centre, which is a maximum when

$$\sin(cl - v) = \pm 1,$$

that is, when the mean anomaly of the moon is either 90° or 270°. Thus, when the moon is in quadrature, the equation of the centre is[22] $\pm 6^\circ\ 17'\ 19''.7$, double the eccentricity of the orbit. In syzigies it is zero.

759. The most remarkable of the periodic inequalities next to the equation of the centre, is the evection[23]

$$4{,}589''.61\sin\left(2l - 2ml - cl + v\right),$$

which is at its maximum and $= \pm 4{,}589''.61$, when $2l - 2ml - cl + v$ is either 90° or 270°, and it is zero when that angle is either 0° or 180°. Its period is found by computing the value of its argument in a given time, and then finding by proportion the time required to describe 360°, or a whole circumference. The synodic motion of the moon in 100 Julian years is

$$445{,}267^{\circ}.1167992 = l - ml$$

and

$$890{,}534^{\circ}.2335984 = 2\{l - ml\}$$

is double the distance of the sun from the moon in 100 Julian years. If $477{,}198^{\circ}.839799$ the anomalistic motion of the moon in the same period be subtracted, the difference $413{,}335^{\circ}.3937994$ will be the angle $2l - 2ml - cl + v$, or the argument of the evection in 100 Julian years: whence

$$413{,}335^{\circ}.3937994 : 360^{\circ} :: 365^{d}.25 : 31^{d}.811939 =$$

the period of the evection. If t be any time elapsed from a given period, as for example, when the evection is zero, the evection may be represented for a short time by

$$4{,}589''.61 \cdot \sin\left\{\frac{360^{\circ}\,.t}{31.811939}\right\}.$$

This inequality is a variation in the equation of the centre, depending on the position of the apsides of the lunar orbit. When the apsides are in syzigies, as in figure 101, the action of the sun increases the eccentricity of the moon's orbit or the equation of the centre. For if the moon be in conjunction at m, the sun draws her from the earth; and if she be in opposition in m', the sun draws the earth from her; in both cases increasing the moon's distance from the earth, and thereby the eccentricity or equation of the centre. When the moon is in any other point of her orbit, the action of the sun may be resolved into two, one in the direction of the tangent, and the other according to the radius vector. The latter increases the moon's gravitation to the earth, and is at its maximum when the moon is in quadratures; as it tends to diminish the distance QE, it makes the ellipse still more eccentric, which increases the equation of the centre. This increase is the evection. Again, if the line of apsides be at right angles to SE, the line joining the centres of the sun and the earth, the action of the sun on the moon at m or m', figure 102, by

fig. 101.

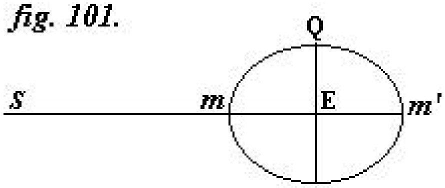
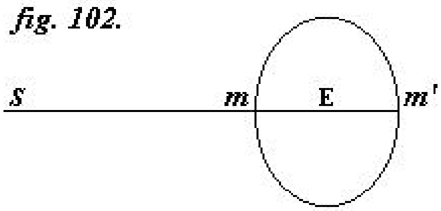


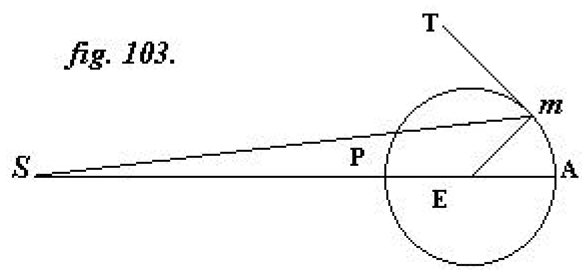
fig. 102.



increasing the distance from the earth, augments the breadth of the orbit, thereby making it approach the circular form, which diminishes the eccentricity. If the moon be in quadratures, the increase in the moon's gravitation diminishes her distance from the earth, which also diminishes the eccentricity, and consequently the equation of the centre. This diminution is the evection. Were the changes in the evection always the same, it would depend on the angular distances of the sun and moon, but its true value varies with the distance of the moon from the perigee of her orbit. The evection was discovered by Ptolemy,[24] in the first century after Christ, but Newton[25] showed on what it depends.

760. The variation is an inequality in the moon's longitude, which increases her velocity before conjunction, and retards her velocity after it. For the sun's force, acting on the moon according to *Sm*, fig. 103, may be resolved into two other forces, one in the direction of *m*E, which produces the evection, and the other in the direction of *m*T, tangent to the lunar orbit. The latter produces the variation which is expressed by

fig. 103.



$$2{,}370''\sin 2\{l - ml\}.$$

This inequality depends on the angular distance of the sun from the moon, and as she runs through her period whilst that distance increases 90°, it must be proportional to the sine of twice the angular distance. Its maximum happens in the octants when $l - ml = 45^\circ$, it is zero when the angular distance of the moon fom the sun is either zero, or when the moon is in quadratures. Thus the variation vanishes in syzigies and quadratures, and is a maximum in the octants.

The angular distance of the moon from the sun depends on its synodic motion: it varies

$$\frac{360^\circ}{29^d.530588}\text{ daily,}$$

and

$$2(l - ml) = \frac{2.360^\circ}{29^d.530588},$$

hence its period is

$$\frac{29^d.530588}{2} = 14^d.765294.$$

Thus the period of the variation is equal to half the moon's synodic revolution. The variation was discovered by Tycho Brahe[26], and was first determined by Newton.[27]

761. The annual equation

$$673''.70\sin\{c'ml - v'\}$$

is another remarkable periodic inequality in the moon's longitude. The action of the sun which produces this inequality is similar to that which causes the acceleration of the moon's mean motion. The annual equation is occasioned by a variation in the sun's distance from the earth, it consequently arises from the eccentricity of the terrestrial orbit. When the sun is in perigee his action is greatest, and he dilates the lunar orbit, so that the angular motion of the moon is diminished; but as the sun approaches the apogee the orbit contracts, and the moon's angular motion is accelerated. This change in the moon's angular velocity is the annual equation. It is a periodic inequality similar to the equation of the centre in the sun's orbit, which retards the motion of the moon when that of the sun increases, and accelerates the motion of the moon when the motion of the sun diminishes, so that the two inequalities have contrary signs.

The period of the annual equation is an anomalistic year. It was discovered by Tycho Brahe[28] by computing the places of the moon for various seasons of the year, and comparing them with observation. He found the observed motion to be slower than the mean motion in the six months employed by the sun in going from perigee to apogee, and the contrary in the other six months. It is evident that as the action of the sun on the moon varies with his distance, and therefore depends on the eccentricity of the earth's orbit, whatever affects the eccentricity must influence all the motions of the moon.

762. The variation has been ascribed to the effect of that part of the sun's force that acts in the direction of the tangent; and the evection to the effect of the part which acts in the direction of the radius vector, and alters the ratio of the perigean and apogean gravities of the moon from that of the inverse squares of the distance. The annual equation does not arise from the direct effect of either, but from an alteration in the mean effect of the sun's disturbing force in the direction of the radius vector which lessens the gravity of the moon to the earth.

763. Although the causes of the lesser inequalities are not so easily traced as those of the four that have been analysed, yet some idea of the sources from whence they arise may be formed by considering that when the moon is in her nodes, she is in the plane of the ecliptic, and the action of the sun being in that plane is resolved into two forces only; one in the direction of the moon's radius vector, and the other in that of the tangent to her orbit. When the moon is in any other part of her orbit, she is either above or below the plane of the ecliptic, and the line joining the sun and moon, which is the direction of the sun's disturbing force, being out of that plane, the sun's force is[29] resolved into three component forces; one in the direction of the moon's radius vector, another in the tangent to her orbit, and the third perpendicular to the plane of her orbit, which affects her latitude. If then the absolute action of the sun be the same in these two positions of the moon, the component forces in the radius vector and tangent must be less than when the moon is in her nodes by the whole action in latitude. Hence any inequality like the evection, whose argument does not depend on the place of the nodes, will be different in these two positions of the moon, and will require a correction, the argument of which should depend on the position of the nodes. This circumstance introduces the inequality

$$54''.83 \cdot \sin\left(2gl - 2l + 2ml - 2q\right)$$

in the moon's longitude. The same cause introduces other inequalities in the moon's longitude, which are the corrections of the variation and annual equation. But the annual equation requires a correction from another cause which will introduce other terms in the perturbations of the moon in longitude; for since it arises from a change in the mean effect of the sun's disturbing force, which diminishes the moon's gravity, its coefficient is computed for a certain value of the moon's gravity, consequently for a given distance of the moon from the earth; hence, when she has a different distance, the annual equation must be corrected to suit that distance.

764. In general, the numerical coefficients of the principal inequalities are computed for particular values of the sun's disturbing force, and of the moon's gravitation; as these are perpetually changing, new inequalities are introduced, which are corrections to the inequalities computed in the first hypothesis. Thus the perturbations are a series of corrections. How far that system is to be carried, depends on the perfection of astronomical instruments, since it is needless to compute quantities that fall within the limits of the errors of observation.

765. When Laplace[30] had determined all the inequalities in the moon's longitude of any magnitude arising from every source of disturbance, he was surprised to find that the mean longitude computed from the tables in Lalande's astronomy[31] for different epochs did not correspond with the mean longitudes computed for the same epochs from the tables of Lahere and Bradley,[32] the difference being as follows:–

Epochs.	Errors.[33]
1766	$-3''$
1779	$9''.3$
1789	$17''.6$
1801	$28''.5$

Whence it was to be presumed that some inequality of a very long period affected the moon's mean motion, which induced him to revise the whole theory of the moon. At last he found that the series which determines the mean longitude contains the term

$$g^2 e'^3 e \cdot \frac{a}{a'} \cdot \frac{\sin\{3v - 3mv + 3c'mv - 2gv - cv + 2\theta + \varpi - 3\varpi'\}}{\{3 - 3m + 3c'm - 2g - c\}^2} = g^2 e'^3 . e \cdot \frac{a}{a'} \cdot \frac{\sin\{2\theta + \varpi - 3\varpi'\}}{\{3 - 3m + 3c'm - 2g - c\}^2}$$

depending on the disturbing action of the sun, that appeared to be the cause of these errors.

The coefficient of this inequality is so small that its effect only becomes sensible in consequence of the divisor

$$\{3 - 3m + 3c'm - 2g - c\}^2$$

acquired from the double integration. Its maximum, deduced from the observations of more than a century, is $15''.4$. Its argument is twice the longitude of the ascending node of the lunar orbit, plus the longitude of the perigee, minus three times the longitude of the sun's perigee, whence its period may be found to be about 184 years.

The discovery of this inequality made it necessary to correct the whole lunar tables.

766. By reversion of series the moon's latitude in functions of her mean motion is found to be[34] [35]

$$
\begin{aligned}
s = &+18{,}539''.8 \cdot \sin(gl - q) \\
&+ \quad 12''.6 \cdot \sin(3gl - 3q) \\
&+ \quad 527''.7 \cdot \sin(2l - 2mv - gl + q) \\
&+ \quad 1''.0 \cdot \sin(2l - 2ml + gl - q) \\
&- \quad 1''.3 \cdot \sin(gl + cl - v - q) \\
&- \quad 14''.4 \cdot \sin(cl - gl - v + q) \\
&+ \quad 1''.8 \cdot \sin(2l - 2ml - gl + cl - v + q) \\
&- \quad 0''.3 \cdot \sin(2l - 2ml + gl - cl + v - q) \\
&- \quad 15''.8 \cdot \sin(2l - 2ml - gl - cl + v + q) \\
&+ \quad 23''.8 \cdot \sin(gl + c'ml - v' - q) \\
&- \quad 25''.1 \cdot \sin(gl - c'ml + v' - q) \\
&- \quad 10''.3 \cdot \sin(2l - 2ml - gl + c'ml - v' + q) \\
&+ \quad 22''.0 \cdot \sin(2l - 2ml - gl - cml + v' + q) \\
&+ \quad 25''.7 \cdot \sin(2cl - gl - 2v + q) \\
&- \quad 5''.4 \cdot \sin(2l - 2ml - 2cl - gl + 2v + q).
\end{aligned}
$$

767. The only inequality in the moon's latitude that was discovered by observation is

$$527''.7 \sin(2l - 2ml - gl + q).$$

Tycho Brahe[36] observed, in comparing the greatest latitude of the moon in different positions with regard to her nodes, that it was not always the same, but oscillated about its mean value of $5^\circ\, 9'$, and as the greatest latitude is the measure of the inclination of the orbit, it was evident that the inclination varied periodically. Its period is a semi-revolution of the sun with regard to the moon's nodes.

768. By reversion of series it will he found that the lunar parallax at the equator in terms of the mean motions is[37]

$$
\begin{aligned}
\frac{1}{r} = &+3{,}420''.89 \\
&+ \quad 186''.48 \cos(cl - v) \\
&+ \quad 28''.54 \cos(2l - 2ml) \\
&+ \quad 34''.43 \cos(2l - 2ml - cl + v) \\
&+ \quad 3''.05 \cos(2l - 2ml + cl - v)
\end{aligned}
$$

$$
\begin{aligned}
&- && 0''.26\cos(2l-2ml+c'ml-v') \\
&+ && 1''.92\cos(2l-2ml-c'ml+v') \\
&- && 0''.32\cos(c'ml-v') \\
&- && 0''.24\cos(2l-2ml-cl+c'ml+v-v') \\
&+ && 1''.45\cos(2l-2ml-cl-c'ml+v+v') \\
&+ && 1''.20\cos(cl-c'ml-v+v') \\
&- && 0''.92\cos(cl+c'ml-v-v') \\
&+ && 10''.24\cos(2cl-2v) \\
&- && 0''.41\cos(2cl-2l+2ml-2v) \\
&+ && 0''.03\cos(2gl-2q) \\
&- && 0''.15\cos(2l-2ml-2gl+2q) \\
&- && 0''.70\cos(cl-2gl-v+2q) \\
&- && 0''.06\cos(2l-2ml-2gl+cl+2q-v) \\
&- && 0''.98\cos(l-ml) \\
&+ && 0''.14\cos(l-ml+c'ml-v') \\
&+ && 0''.18\cos(2l-2ml+cl-c'ml-v+v') \\
&+ && 0''.57\cos(4l-4ml-cl+v) \\
&+ && 0''.40\cos(4l-4ml-2cl+2v) \\
&- && 0''.03\cos(2l-2ml-2cl-c'ml+2v+v') \\
&+ && 0''.14\cos(2cl+2l-2ml-2v).
\end{aligned}
$$

769. The planets are at so great a distance from the sun, and from one another, that their form has no perceptible effect on their mutual motions; and, considered as spheres, their action is the same as if their mass were united in their centre of gravity: but the satellites are so near their respective planets that the ellipticity of the latter has a considerable influence on the motions of the former. This is particularly evident in the moon, whose motions are troubled by the spheroidal form of the earth.

Notes

[1] The terms $+c'mv$ and $-c'mv$ in the 11th and 12th lines are unaccented in the 1st edition. However, the equivalent terms in equation (239) are accented.

[2] See note 6, *Bk. III, Chap. I.*

[3] France. Bureau des longitudes, *Tables astronomiques par M. Burg / publiees par le bureau des longitudes de France*, Paris : Chez Courcier, 1806.

[4] Burckhardt, Johann Carl, *Tables astronomiques, publiees par le Bureau des Longitudes de France: tables de la lune, par M. Burckhardt*, Paris : Veuve Courcier, 1812.
[5] See note 55, *Bk. II, Chap. VI.*
[6] See note 4, *Introduction.*
[7] See note 55, *Preliminary Introduction.*
[8] See note 5.
[9] See note 15, *Preliminary Dissertation.*
[10] See note 54, *Preliminary Dissertation.*
[11] Bouvard, Alexis, 1767-1843, astronomer, born in Contamines, France. Bouvard was Director of the Paris Observatory and discovered eight comets. He wrote the *Tables astronomiques* of Jupiter and Saturn in 1808 and of Uranus in 1821. The eventual failure of the Uranus tables led him to speculate later that the errors were due to disturbances from another celestial body. Mary Somerville offered a similar interpretation in the 6th edition of her *On the Connexion of the Physical Sciences* in 1842. Bouvard died three years before the discovery of Neptune in 1846. (see also note 48, *Bk. I Foreword*; notes 28 & 39, *Bk. II Foreword*; note 38, *Bk. II Foreword*; *note 38, Bk. II Chap. XIV*; and *Foreword to the Second Edition.*)
[12] See note 49, *Preliminary Dissertation.*
[13] The equation *reads* $\frac{3}{2}m\int\left(e'^2-\overline{e}^2\right)dv$ in the 1st edition (published erratum).
[14] Clairaut, Alexis Claude, 1713-1765, mathematician, born in Paris, France. Clairaut translated Newton's *Principia* into French, but later when working on the three-body problem (1745), he concluded that Newton's inverse square law was incorrect. By 1748 he had retracted this conclusion and published his new results in *Théorie de la lune* in 1752. Clairaut later used his techniques to calculate the return of Halley's comet in 1759. When the comet returned to perihelion only a month before his predicted date Clairaut became a public hero.
[15] Buffon, Georges-Louis Leclerc, comte de, 1707-1788, naturalist, born in Montbard, France. In 1739 he was made director of the Jardin du Roi, and wrote his *Histoire naturelle* (1749-67, Natural History). (see also note 56, *Preliminary Dissertation.*)
[16] See note 6, *Bk. III, Chap. I.*
[17] See note 32, *Preliminary Dissertation.*
[18] The coefficient of the 2nd term *reads* $C_{/}$ in the 1st edition.
[19] See note 6, *Bk. III, Chap. I.*
[20] See note 6, *Bk. III, Chap. I.*
[21] The parentheses in the argument of sine are mismatched in the 1st edition.
[22] This value is punctuated with a period and *reads* $\pm 6^{\circ}\ 17'\ 19''.7.$ in the 1st edition.
[23] The parentheses in the argument of sine are mismatched in the 1st edition.
[24] See note 15, *Preliminary Dissertation.*
[25] See note 1, *Preliminary Dissertation.*
[26] See note 6, *Bk. II, Chap. I.*
[27] See note 1, *Preliminary Dissertation.*
[28] See note 6, *Bk. II, Chap. I.*
[29] The "i" is missing in "is" in the 1st edition.
[30] See note 4, *Introduction.*
[31] Lalande, Joseph Jérôme Le Français de, 1732-1807, astronomer, born in Bourg-en-Bresse, France. He was professor of Astronomy in the Collège de France, and later director of the Paris Observatory. He determined the lunar parallax in 1751. His major work is *Traité d'astronomie* (1764). He wrote his *Histoire céleste française* in 1801 and produced a comprehensive star catalogue in the same year.
[32] See note 38, *Preliminary Dissertation.*
[33] The 1801 error in the table *reads* $28''\ 5$ (missing decimal) in the 1st edition.
[34] The parentheses in the arguments of the last 13 terms are miss-matched and *read* {argument) in the 1st edition.
[35] The period is omitted at the end of this expression in the 1st edition.
[36] See note 6, *Bk. II, Chap. I.*
[37] The coefficient of the 23rd term *reads* $+0''.4$ in the 1st edition. If this is a misprint it perhaps ought to read $+0''.04$.