BOOK III

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CHAPTER III

INEQUALITIES FROM THE FORM OF THE EARTH

770. THE attraction of the disturbing matter is equal to the sum of all the molecules in the excess of the terrestrial spheroid above a sphere whose radius is half the axis of rotation, each molecule being divided by its distance from the moon; and the finite values of this action, after it has been resolved in the direction of the three co-ordinates of the moon, are the perturbations in longitude, latitude, and distance, caused by the non-sphericity of the earth. In the determination of these inequalities, therefore, results must be anticipated that can only be obtained from the theory of the attraction of spheroids. By that theory it is found that if r be the ellipticity of the earth, R` its mean radius, f the ratio of the centrifugal force at the equator to gravity, and n the sine of the moon's declination, the attraction of the redundant matter at the terrestrial equator is

$$\left(\tfrac{1}{2}f-r\right)\frac{R`}{r^3}\left(n-\tfrac{1}{3}\right)$$

the sum of the masses of the earth and moon being equal to unity. Hence the quantity R, which expresses the disturbing forces of the moon in equation (208) must be augmented by the preceding expression.

771. By spherical trigonometry n, the sine of the moon's declination in functions of her latitude and longitude, is

$$n=\sin w\sqrt{1-s^2}\sin fv+s\cos w\ ,$$

in which w is the obliquity of the ecliptic, s the tangent of the moon's latitude, and fv her true longitude, estimated from the equinox of spring. The part of the disturbing force R that depends on the action of the sun, has the form Qr^2 when the terms depending on the solar parallax are rejected. Hence

$$R=Qr^2-\left(r-\tfrac{1}{2}f\right)\cdot\frac{R`^2}{r^3}\left(\sin^2 w\,.\sin^2 fv+2s\sin w\,.\cos w\,.\sin fv\right)$$

very nearly; but $s=g\sin\left(gv-q\right)$ by article 696, and if

$$\frac{1}{a^3}\text{ be put for }\frac{1}{r^3}$$

[then]

$$R = Qr^2 - \left(r - \tfrac{1}{2}f\right)\cdot\frac{R'^2}{a^3}\sin w\,.\cos w\,.g\cos\left(gv - fv - q\right);$$

when all terms are rejected except those depending on the angle $gv - fv - q$, which alone have a sensible effect in troubling the motion of the moon.

772. If this force be resolved in the direction of the three co-ordinates of the moon, and the resulting values of

$$\frac{dR}{du} \quad \frac{dR}{dv} \quad \frac{dR}{ds}$$

substituted in the equations in article 695, they will determine the effect which the form of the earth has in troubling the motions of that body. But the same inequalities are obtained directly and with more simplicity from the differential of the periodic variation of the epoch in article 439, which, in neglecting the eccentricity of the lunar orbit, becomes

$$d\epsilon = -2a^2\left(\frac{dR}{da}\right)ndt\,.$$

Now

$$2a^2\left(\frac{dR}{da}\right) = 4ar^2Q + 6\left(r - \tfrac{1}{2}f\right)\cdot\frac{R'^2}{a^2}\cdot\sin w\,.\cos w\,.g\cos\left(gv - fv - q\right).$$

But by article 438 the variation of dR is zero, consequently the coefficient of $\cos\left(gv - fv - q\right)$ must be zero in R. Then if $d\,.r^2Q$ be the part of r^2Q that depends on the compression of the earth,

$$0 = d\,.r^2Q - \left(ar - \tfrac{1}{2}af\right)\cdot\frac{R'^2}{a^3}\cdot\sin w\,.\cos w\,.g\cos\left(gv - fv - q\right),$$

and eliminating $d\,r^2Q$,

$$2da^2\left(\frac{dR}{da}\right) = 10\left(r - \tfrac{1}{2}f\right)\frac{R'^2}{a^3}\cdot\sin w\,.\cos w\,.g\cos\left(gv - fv - q\right).$$

And if dv be put for ndt,

$$d\epsilon = -10\left(r - \tfrac{1}{2}f\right)\cdot\frac{R'^2}{a^3}\cdot\sin w\,.\cos w\,.g\,dv\,.\cos\left(gv - fv - q\right).$$

This equation is referred to the plane of the lunar orbit, but in order to reduce it to the plane of the ecliptic the equation (154) must be resumed, which is

$$dv_{/} = dv\left(1 + \tfrac{1}{2}s^2 - \tfrac{1}{2}\frac{ds^2}{dv^2}\right)$$

$dv_{/}$ being the arc dv projected on the plane of the ecliptic, or fixed plane. By article 436

$$s = q\sin fv - p\cos fv,$$

whence

$$\frac{ds}{dv} = \left(fq - \frac{dp}{dv}\right)\cos fv + \left(fp + \frac{dq}{dv}\right)\sin fv + \&c.$$

and neglecting periodic quantities depending on fv,

$$dv_{/} = dv + \frac{qdp - pdq}{2}, \text{ very nearly}.$$

Hence, in order to have $d.dv_{/}$ it is only necessary to add

$$\frac{qdp - pdq}{2} \text{ to } d.dv.$$

For the same reason the angle $d\in$ will be projected on the plane of the ecliptic if $\frac{qdp - pdq}{2}$ be added to it, so that

$$d\in_{/} = d\in + \frac{qdp - pdq}{2}.$$

Now $s = g\sin(gv - q)$ may be put under the form

$$s = g\cos(gv - fv - q)\sin fv + g\sin(gv - fv - q)\cos fv,$$

and comparing it with

$$s = q\sin fv - p\cos fv,$$

the result is

$$p = -g\sin(gv - fv - q) \quad q = g\cos(gv - fv - q),$$

whence

$$dp = -(g - f)qdv$$
$$dq = +(g - f)pdv$$

$$R = r^2 Q - \left(r - \tfrac{1}{2} f \right) \frac{R'^2}{a^2} \sin w \cos w \,.\, q,$$

and

$$\frac{dR}{dq} = -\left(r - \tfrac{1}{2} f \right) \frac{R'^2}{a^2} \sin w \cos w;$$

in consequence of this the values of *dp*, *dq*, in article 439, become

$$dp = -(g - f)\, q dv + \left(r - \tfrac{1}{2} f \right) \frac{R'^2}{a^2} \sin w \cos w \cdot dv$$
$$dq = +(g - f)\, p dv;$$

therefore *dp* contains the term

$$\frac{\left(r - \tfrac{1}{2} f \right)}{g - f} \cdot \frac{R'^2}{a^2} \sin w \cos w \,.\, dv\,;$$

and as $ds = dq \,.\sin fv - dp \cos fv,$ the latitude of the moon is subject to the inequality

$$-\frac{\left(r - \tfrac{1}{2} f \right)}{g - f} \cdot \frac{R'^2}{a^2} \sin w \cos w g \sin fv. \qquad \textbf{(243)}$$

773. The constant part of q produces in $\dfrac{qdp - pdq}{2}$ the term[1]

$$+\tfrac{1}{2}\left(r - \tfrac{1}{2} f \right) \frac{R'^2}{a^2} \cdot \sin w \cos w \,.\, g \cos(gv - fv - q);$$

whence $d\in_{/}$, which is the value of $d\in$ when referred to the plane of the ecliptic, becomes

$$d\in = -\frac{19}{2} \cdot \left(r - \tfrac{1}{2} f \right) \frac{R'^2}{a^2} \sin w \cos w \,.\, g \cos(gv - fv - q) \,.\, dv\,,$$

which gives in $\in$, and consequently in the true longitude of the moon the inequality

$$-\frac{19}{2} \cdot \frac{\left(r - \tfrac{1}{2} f \right)}{g - f} \cdot \frac{R'^2}{a^2} \cdot \sin w \cos w \,.\, g \sin(gv - fv - q). \qquad \textbf{(244)}$$

774. Now,

$$\frac{R'}{a} = 0.0165695, \qquad w = 23^\circ\ 28', \ \&c.$$

$$f = \frac{1}{289}, \quad g = 0.0900684, \quad g = 1.00402175,$$

and the argument $gv - fv - q$ is the mean longitude of the moon. Thus every quantity is known, except r, the compression, which may therefore be determined by comparing the coefficient, computed with these data, with the coefficient of the same inequality given by observation. By Burg's Tables,[2] it is $-6''.8$, and by Burckhardt's,[3] $-7''.0$. The mean of these $-6''.9$ gives the compression

$$r = \frac{1}{303.22}.$$

By the theory of the rotation of spheroids, it is found that if the earth be homogeneous, the compression is $\frac{1}{230}$. Consequently the earth is of variable density.

That the inequalities of the moon should disclose the interior structure of the earth, is a singular instance of the power of analysis.

775. The inequality in the moon's latitude, depending on the same cause, confirms these results. Its coefficient, determined by Burg[4] and Burckhardt[5] from the combined observations of Maskelyne[6] and Bradley,[7] is $-8''.0$, which, compared with the coefficient of

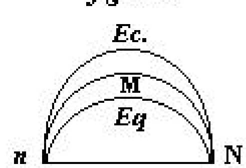
$$-\frac{\left(r - \frac{1}{2}f\right)}{g - f} \cdot \frac{R^2}{a^2} \sin w \cos w \,. g \sin fv,$$

computed with the preceding data, gives $\frac{1}{305.26}$ for the compression, which also proves that the earth is not homogeneous.

776. Since the coefficients of both inequalities are greater in supposing the earth to be homogeneous, it affords another proof that the gravitation of the moon to the earth is composed of the attraction of all its particles. Thus the eclipses of the moon in the early ages of astronomy showed the earth to be spherical, and her motions, when perfectly known, determine its deviation from that figure. The ellipticity of the earth, obtained from the motions of the moon, being independent of the irregularities of its form, has an advantage over that deduced from observations with the pendulum, and from the arcs of the meridian.

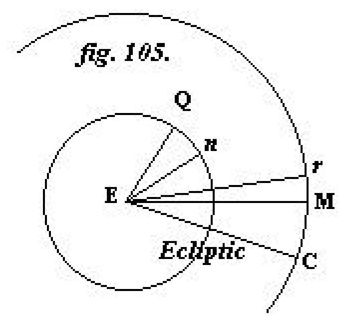
777. The inequality in the moon's latitude, arising from the ellipticity of the earth, may be represented by supposing that the orbit of the moon, in place of moving with the earth on the plane of the ecliptic, and preserving the same inclination of $5^\circ\ 9'$ to that plane, moves with a constant inclination of $8''$ on a plane NM*n* passing between the ecliptic and

fig. 104.



the equator, and through *n*N, the line of the equinoxes. The inequality in question diminishes the inclination of the lunar orbit to the ecliptic, when its ascending node coincides with the equinox of spring; it augments it when this node coincides with the autumnal equinox.

778. This inequality is the re-action of the nutation in the terrestrial axis, discovered by Bradley;[8] hence there would be equilibrium round the centre of gravity of the earth, in consequence of the forces which produce the terrestrial nutation and this inequality in the moon's latitude, if all the molecules of the earth and moon were fixedly united by means of a lever; the moon compensating the smallness of the force which acts on her by the length of the lever to which she is attached, for the distance of the common centre of gravity of the earth and moon from the centre of the earth is less than the earth's semidiameter.



The proof of this depends on the rotation of the earth; but some idea may be formed of this re-action from the annexed diagram. Let EC be the plane of the ecliptic, seen edgewise; Q the earth's equator; E its centre, and M the moon. Then QEC is the obliquity of the ecliptic, and MEC the latitude of the moon.

The moon, by her action on the redundant matter at Q, draws the equator to some point *n* nearer to the ecliptic, producing the nutation QE*n*; but as re-action is equal and contrary to action, the matter at Q draws the moon from M to some point *r*, thereby producing the inequality ME*r* in her latitude, that has been determined. Laplace[9] finds the analytical expressions of the areas ME*r* and QE*n*, and thence their moments; the one from the preceding inequality in the moon's latitude, the other from the formulae of nutation in the axis of the earth's rotation from the direct action of the moon. These two expressions are identical, but with contrary signs, proving them, as he supposed, to be the effects of the direct and reflected action of the moon.

779. The form of the earth increases the motion of the lunar nodes and perigee by $0.000000085484v$, an insensible quantity. The ellipticity of the lunar spheroid has no perceptible effect on her motion.

Notes

[1] The factor $\frac{R^{-2}}{a^2}$ *reads* $\frac{R^2}{a^2}$ in the 1st edition.
[2] See note 3, *Bk. III, Chap. II.*
[3] See note 4, *Bk. III, Chap. II.*
[4] See note 2.
[5] See note 3.
[6] See note 55, *Bk. II, Chap. VI.*
[7] See note 38, *Preliminary Dissertation.*
[8] See note 7.
[9] See note 4, *Introduction.*