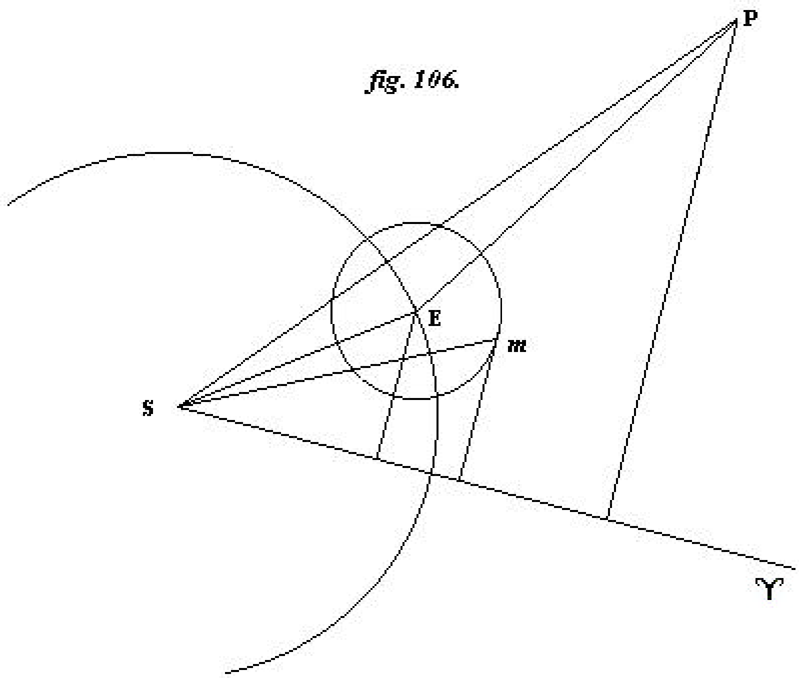
BOOK III

CHAPTER IV

INEQUALITIES FROM THE ACTION OF THE PLANETS

780. THE action of the planets produces three different kinds of inequalities in the motions of the moon. The first, and by far the greatest, arising from their influence on the eccentricity of the earth's orbit, which is the cause of the secular inequalities in the mean motion, in the perigee, and nodes of the lunar orbit. The other two are periodic inequalities in the moon's longitude; one from the direct action of the planets on the moon, the other from the perturbations they occasion in the longitude and radius vector of the earth, which are reflected back to the moon by means of the sun.



For, let *S* be the sun, E and *m* the earth and moon, P a planet, and *g* the first point of Aries: then, if *P* be the mass of the planet, its direct action on the moon is $\frac{P}{(Pm)^2}$, which alters the position of the moon with regard to the earth. Again, the disturbing action of the planet on the earth is $\frac{P}{(PE)^2}$, which changes the position of the earth with regard to the moon, in each case producing inequalities of the same order. The latter become sensible from the very small divisors they acquire by integration.

The direct action will be determined first.

If $X`$, $Y`$, $Z`$, x, y, z, be the co-ordinates of the planet and moon, referred to the centre of the earth, and f the distance of the planet from this centre, then

$$f = \sqrt{(X`-x)^2 + (Y`-y)^2 + (Z`-z)^2} .$$

But if X', Y', Z', x', y', z', be the co-ordinates of the planet and the earth referred to the centre of the sun,

$$X` = X' - x', \quad Y` = Y' - y', \quad Z` = Z' - z';$$

and

$$f = \sqrt{\left(X' - (x' + x)\right)^2 + \left(Y' - (y' + y)\right)^2 + \left(Z' - (z' + z)\right)^2};$$

and the attraction of the planet on the moon is

$$\frac{P}{f} - \frac{\frac{1}{2}Pr^2}{f^3} + \frac{3}{2}P\frac{\left(X'x + Y'y + Z'z - xx' - yy' - zz'\right)^2}{f^5} + \&\text{c.}$$

The ecliptic being the fixed plane,

$$z' = 0, \quad r' = \frac{1}{u'}, \quad v' = g\,SE.$$

Then, if $R_/ = SP$, $U = g\,SP$, and S, be the radius vector, longitude, and heliocentric latitude of the planet, it is evident that

$$x' = \frac{\cos v'}{u'}, \qquad y' = \frac{\sin v'}{u'}, \qquad r' = \frac{\sqrt{1+s^2}}{u'},$$

$$X' = R_/ \cos U, \quad Y' = R_/ \sin U, \quad Z' = R_/ S;$$

hence

$$f = \sqrt{R^2\left(1+S^2\right) + r'^2 - 2Rr'\cos\left(U - v'\right)};$$

therefore the action of the planet on the moon is

$$\frac{P}{f} - \frac{\frac{1}{2}P\left(1+s^2\right)}{u^2 f^3} + \frac{3}{2}P\frac{\left(R_/\cos(v-U) - r\cos(v-v') + R_/ s S\right)^2}{u^2 f^5} + \&\text{c.}$$

or, omitting S^2, it is

$$\frac{P}{f} + \frac{P\left(1-2s^2\right)}{4u^2 f^3} + \&\text{c. }\&\text{c.}$$

The first term does not contain the co-ordinates of the moon, and therefore does not affect her motion; and the only term of the remainder of the series that has a sensible influence is $\frac{P}{4u^2 f^3}$, which, therefore, forms a part of R in (208); and, with regard to the action of the planets alone, $R = \frac{P}{4u^2 f^3}$. But, by article 446, the development of f is

$$\tfrac{1}{2}A^{(0)}+A^{(1)}\cos\left(U-v'\right)+A^{(2)}\cos 2\left(U-v'\right)+\&\text{c.}$$

If i be the ratio of the mean motion of the planet to that of the moon, by equation (212)

$$U=iv-2ie\sin\left(cv-v\right)+\&\text{c.}$$

Hence, if iv be put for U, and mv for v', it is evident that

$$R=\frac{P}{4u^2}\left\{\tfrac{1}{2}A^{(0)}+A^{(1)}\cos\left(i-m\right)v+A^{(2)}\cos 2\left(i-m\right)v+\&\text{c.}\right\}$$

The only term of the parallax in which this value of R is sensible is

$$-\frac{1}{h^2}\left(\frac{dR}{du}\right)$$

which becomes

$$\frac{PA^{(0)}}{4h^2u^3}+\frac{P}{2h^2u^3}\left\{A^{(1)}\cos\left(i-m\right)v+A^{(2)}\cos 2\left(i-m\right)v+\&\text{c.}\right\};$$

or, if e^2 and g^2 be neglected, $u^{-3}=a^3$, and the periodic part of $-\frac{1}{h^2}\left(\frac{dR}{du}\right)$ is

$$\frac{Pa^3}{2h^2}\left\{A^{(1)}\cos\left(i-m\right)v+A^{(2)}\cos 2\left(i-m\right)v+\&\text{c.}\right\}$$

But, by the second of equations (209), $-\frac{1}{h^2}\left(\frac{dR}{du}\right)$ contains the variation of $\frac{m'u'^3}{2h^2u^3}$ which is

$$-\frac{3m'u'^3}{2h^2u^4}\delta u=-\frac{3m^2}{2}\cdot\delta u\ .$$

Let

$$\delta u=G_1\cos\left(i-m\right)v+G_2\cos 2\left(i-m\right)v+G_3\cos 3\left(i-m\right)v+\&\text{c.}$$

Therefore the direct disturbance of the planets gives

$$\frac{d^2u}{dv^2}+u=-\frac{Pa^3}{2}\left\{A_1\cos\left(i-m\right)v+A_2\cos 2\left(i-m\right)v+\&\text{c.}\right\}$$
$$+\frac{3m^2}{2}\left\{G_1\cos\left(i-m\right)v+G_2\cos 2\left(i-m\right)v+\&\text{c.}\right\}=$$
$$G_1\left(1-\left(i-m\right)^2\right)\cos\left(i-m\right)v+G_2\left(1-4\left(i-m\right)^2\right)\cos 2\left(i-m\right)v+\&\text{c.}$$

And comparing similar cosines,[1]

$$G_1 = -\frac{\frac{1}{2}P.\,A_1.a^3}{1-\frac{3}{2}m^2-(1-m)^2}$$
$$G_2 = -\frac{\frac{1}{2}P.\,A_2.a^3}{1-\frac{3}{2}m^2-4(1-m)^2}$$
$$G_3 = -\frac{\frac{1}{2}P.\,A_3.a^3}{1-\frac{3}{2}m^2-9(1-m)^2}$$
$$\&c. \qquad \&c.$$

and thus the integral u or (228) acquires the term

$$-\tfrac{1}{2}Pa^3\left\{\frac{A_1\cos(i-m)v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{A_2\cos 2(i-m)v}{1-\frac{3}{2}m^2-4(i-m)^2}+\&c.\right\}$$

consequently, the mean longitude $nt+\in$ contains the term

$$\frac{Pa^3}{i-m}\left\{\frac{A_1\sin(i-m)v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{\frac{1}{2}A_2\sin 2(i-m)v}{1-\frac{3}{2}m^2-4(i-m)^2}+\&c.\right\}$$

or if a^3 be eliminated by $\frac{m'a^3}{a'^3}=m^2$

$$\frac{\frac{P}{m'}m^2a'^3}{i-m}\left\{\frac{A_1\sin(i-m)v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{\frac{1}{2}A_2\sin 2(i-m)v}{1-\frac{3}{2}m^2-4(i-m)^2}+\&c.\right\} \qquad \textbf{(245)}$$

m' being the mass of the sun.

If B_1, B_2, &c., be put for A_1, A_2, &c., it becomes

$$\frac{\frac{P}{m'}m^2a^3}{i-m}\left\{\frac{B_1\sin(i-m)v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{\frac{1}{2}B_2\sin 2(i-m)v}{1-\frac{3}{2}m^2-4(i-m)^2}+\&c.\right\} \qquad \textbf{(246)}$$

which is the inequality in the moon's mean longitude, arising from the action of a planet inferior to the earth.

And if a be the ratio of the mean distance of the planet from the sun to that of the sun from the earth, the substitution of $a^3 B_1,\ a^3 B_2$, &c., for A_1, A_2, &c., in equation (245), gives[2]

$$\frac{\frac{P}{m'} m^2 \cdot a'^3 a^3}{i-m}\left\{\frac{B_1 \sin(i-m)v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{\frac{1}{2}B_2 \sin 2(i-m)v}{1-\frac{3}{2}m^2-4(i-m)^2}+\&\text{c.}\right\} \qquad \textbf{(247)}$$

for the action of a superior planet on the mean longitude of the moon.

781. Besides these disturbances, which are occasioned by the direct action of the planets on the moon, there are others of the same order caused by the perturbations in the radius vector of the earth. The variation of u' was omitted in the development of the coordinates of the moon, but

$$d\cdot\frac{m'u'^3}{2h^2u^3}=\frac{3m'u'^2}{2h^2u^3}du'$$

and when the eccentricities are omitted,

$$h^2=a,\ \text{and}\ \frac{m'a^3}{a'^2}=a'm^2.$$

So

$$d\,\frac{m'u'^3}{2h^2u^3}=\frac{3a'm^2}{2a}du';$$

since $du'=\frac{dr'}{a'}$ are the periodic inequalities in the radius vector of the earth produced by the action of a planet, they are given in (158), and may be represented by

$$a'du'=-\frac{P}{m'}\{K_1\cos(i-m)v+K_2\cos 2(i-m)v+\&\text{c.}\}$$

where the coefficients K_1, K_2, &c. are known, and $(i-m)v$ is the mean longitude of the planet minus that of the earth. Thus

$$\frac{3a'm^2}{2a}du'=-\frac{3m^2}{2}\cdot\frac{P}{m'}\{K_1\cos(i-m)v+K_2\cos 2(i-m)v+\&\text{c.}\}$$

By the method of indeterminate coefficients, it will be found that adu contains the function

$$\frac{3m^2}{2}\cdot\frac{P}{m'}\left\{\frac{K_1\cos(i-m)v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{K_2\cos 2(i-m)v}{1-\frac{3}{2}m^2-4(i-m)^2}+\&\text{c.}\right\}$$

and the mean longitude of the moon is subject to the inequality

$$-\frac{3m^2}{i-m}\cdot\frac{P}{m'}\left\{\frac{K_1\cos(i-m)v}{1-\frac{3}{2}m^2-(i-m)^2}+\frac{K_2\cos 2(i-m)v}{1-\frac{3}{2}m^2-4(i-m)^2}+\&c.\right\} \qquad \textbf{(248)}$$

Numerical Values of the Lunar Inequalities occasioned by the Action of the Planets

782. With regard to the action of Venus, the data in articles 611 and 610 give

$$a = 0.7233325; \quad i-m = 0.04679$$

and

$$\frac{P}{m'} = \frac{1}{356{,}632};$$

hence because

$$a'^3B_1 = 8.872894,$$
$$a'^3B_2 = 7.386580,$$
$$a'^3B_3 = 5.953940,$$

function (246) becomes

$$+0''.62015\sin(i-m)v + 0''.25990\sin 2(i-m)v + 0''.14125\sin 3(i-m)v$$

which is the direct action of Venus on the moon. Now $d\,r' = -\frac{du'}{u'^2}$, and when the eccentricity is omitted, $u'^2 = \frac{1}{a'^2}$; hence $\frac{dr'}{a'} = -a'du'$. But if the action of Venus on the radius vector of the earth be computed by the formula (158), it will be found that

$$\begin{aligned} a'du' = &+0.0000064475\cos(i-m)v \\ &-0.0000184164\cos 2(i-m)v \\ &+0.000002908\cos 3(i-m)v. \end{aligned}$$

This gives the numerical values of the coefficients K^0, K^1, &c.; hence formula (248) becomes

$$\begin{aligned} &+0''.482200\sin(i-m)v \\ &-0''.69336\sin 2(i-m)v \\ &-0''.07380\sin 3(i-v)v, \end{aligned}$$

which is the indirect action of the planets on the moon's longitude. Added to the preceding the sum is

$$
\begin{aligned}
&+1''.10235 \cdot \sin(i-m)v \\
&-0''.43336 \cdot \sin 2(i-m)v \\
&+0''.06745 \cdot \sin 3(i-m)v,
\end{aligned}
$$

the whole action of Venus on the moon's mean longitude.

783. Relative to Mars:

$$
\begin{aligned}
a &= 0.65630030 \\
a'^3 B_1 &= 5.727893 \\
a'^3 B_2 &= 4.404530 \\
a'^3 B_3 &= 3.255964 \\
i-m &= -0.0350306 \\
\frac{P}{m} &= \frac{1}{1{,}846{,}082}.
\end{aligned}
$$

and by formula (158) with regard to Mars,

$$
\begin{aligned}
a'du' = &+0''.00000017778 \ \cos(i-m)v \\
&+0''.0000026121 \quad \cos 2(i-m)v \\
&+0''.000000111 \quad \cos 3(i-m)v;
\end{aligned}
$$

whence the action of Mars on the moon's mean longitude, both direct and indirect, is

$$
\begin{aligned}
&+0''.025583 \cdot \sin(i-m)v \\
&+0''.389283 \cdot \sin 2(i-m)v \\
&-0''.027337 \cdot \sin 3(i-m)v.
\end{aligned}
$$

784. With regard to Jupiter,

$$
\begin{aligned}
a &= 0.192205 \\
a'^3 B_1 &= 0.618817 \\
a'^3 B_2 &= 0.147980 \\
a'^3 B_3 &= 0.0331045 \\
i-m &= -0.06849523
\end{aligned}
$$

$$\frac{P}{m} = \frac{1}{1{,}067.09}.$$

And formula (158) gives for the action of Jupiter on the radius vector of the earth,

$$\begin{aligned} a'du' = &- 0.0000159055 \quad \cos(i-m)v \\ &- 0.0000090791 \quad \cos 2(i-m)v \\ &- 0.00000064764 \cos 3(i-m)v. \end{aligned}$$

Whence it is easy to see that the whole action of Jupiter on the mean longitude of the moon, both direct and indirect, is

$$\begin{aligned} &+0''.74435 . \sin(i-m)v \\ &-0''.24440 . \sin 2(i-m)v \\ &-0''.01282 . \sin 3(i-m)v. \end{aligned}$$

If all these inequalities, resulting from the action of the planets on the moon, be taken with a contrary sign, we shall have the inequalities that this action produces in the expression of the true longitude of the moon, $(i-m)v$ being supposed equal to the mean motion of the planet minus that of the earth.

785. The secular action of the planets on the moon, and the elements of her orbit, may be determined from the term $\frac{PA^0}{4h^2u^3}$; but as it is insensible, the investigation may be omitted.

Notes

[1] The left hand side of the 2nd equation *reads* G^2 in the 1st edition.

[2] The numerator in the 2nd term of equation (247) *reads* $\frac{1}{2}B_2 \sin(i-m)v$ in the 1st edition.