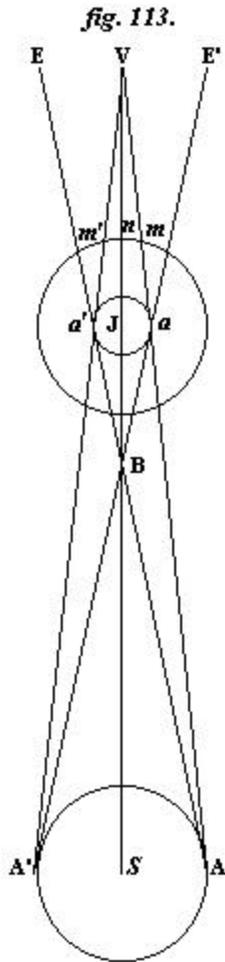


BOOK IV

CHAPTER IV¹

ECLIPSES OF JUPITER'S SATELLITES

918. JUPITER throws a shadow behind him relatively to the sun, in which the three first satellites are always immersed at their conjunctions, on account of their orbits being nearly in the plane of Jupiter's equator; but the greater inclination of the orbit of the fourth, together with its distance, render its eclipses less frequent.



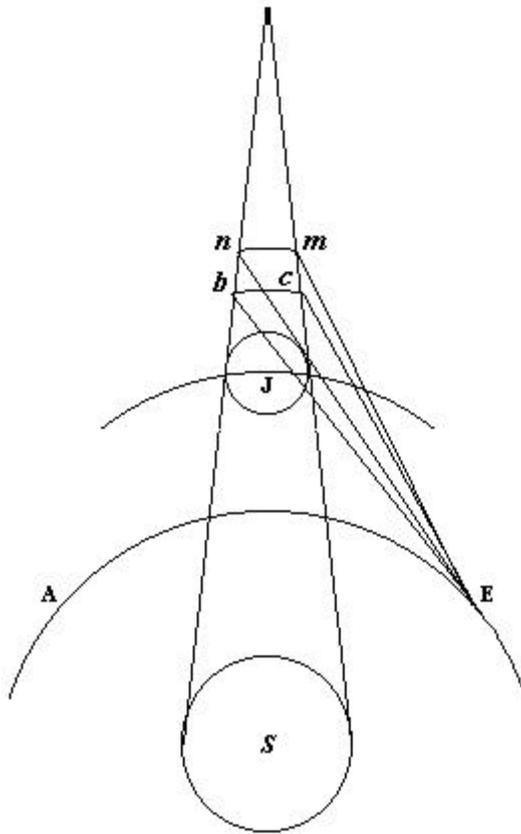
919. Let S and J , fig. 113, be sections of the sun and Jupiter, and mn the orbit of a satellite. Let AE , $A'E'$ touch the sections internally, and AV , $A'V$ externally. If these lines be conceived to revolve about SJV they will form two cones, aVa' and EBE' . The sun's light will be excluded from every part of the cone aVa' , and the spaces $Ea'V$, $E'aV$ will be the penumbra, from which the light of part of the sun will be excluded; less of it will be visible near aV , $a'V$, than near aE' , $a'E$.

920. As the satellites are only luminous by reflecting the sun's rays, they will suddenly disappear when they immerse into the shadow, and they will reappear on the other side of the shadow after a certain time. The duration of the eclipse will depend on the form and size of the cone, which itself depends on the figure of Jupiter, and his distance from the sun.

921. If the orbits of the satellites were in the plane of Jupiter's orbit, they would pass through the axis of the cone at each eclipse, and at the instant of heliocentric conjunction, the sun, Jupiter and the satellite would be on the axis of the cone, and the duration of the eclipses would always be the same, if the orbit were circular. But as all the orbits are more or less inclined to the plane of Jupiter's orbit, the duration of the eclipses varies. If the conjunction happened in the node, the eclipse would still be central; but at a certain distance from the node, the orbit of the satellite would no longer pass through the centre of the cone of the shadow, and the satellite would describe a chord more or less great, but always less than the diameter; hence the duration is variable. The longest eclipses will be those that happen in the nodes, whose position they will determine: the shortest will be observed in the limit or point farthest from the node at which an eclipse can take place, and will consequently determine the inclination of the orbit of that of Jupiter. With the inclination and the node, it will always be possible to compute the duration of the eclipse, its beginning and end.

922. The radius vector of Jupiter makes an angle SJE , fig. 114, with his distance from the earth, varying from 0° to 12° , which is the cause of great variations in the distances at which the eclipses take place, and the phenomena they exhibit.

fig. 114.



923. The third and fourth satellites always, and the second sometimes disappear and reappear on the same side of Jupiter, for if S be the sun, E the earth, and m the third or fourth satellite, the immersion and emersion, are seen in the directions Em, En ; only the immersions or emersions of the first satellite are visible according to the position of the earth; for if ab be the orbit of the first satellite, before the opposition of Jupiter, the immersion is seen in the direction Ea , but the emersion in the direction Eb is hid by Jupiter. On the contrary when the earth is in A , after the opposition of Jupiter, the emersion is seen, and not the immersion; it sometimes happens, that neither of the phases of the eclipses of the first satellite are seen. Before the opposition of Jupiter the eclipses happen on the west side of the planet, and after opposition on the east. The same satellite disappears at different distances from the primary according to the relative positions of the sun, the earth, and Jupiter, but they vanish close to the disc of Jupiter when he is near opposition. The eclipses only happen when the satellites are moving towards the east, the transits only when they are moving towards

the west; their motion round Jupiter must therefore be from west to east, or according to the order of the signs. The transits are real eclipses of Jupiter by his moons, which appear like black spots passing over his disc.

924. It is important to determine with precision the time of the disappearance of a satellite, which is however rendered difficult by the concurrence of circumstances: a satellite disappears before it is entirely plunged in the shadow of Jupiter; its light is obscured by the penumbra: its disc, immersing into the shadow, becomes invisible to us before it is totally eclipsed, its edge being still at a little distance from the shadow of Jupiter, although we cease to see it. With regard to this circumstance, the different satellites vary, since it depends on their apparent distance from Jupiter, whose splendour weakens their light, and makes them more difficult to be seen at the instant of immersion. It also depends on the greater or less aptitude of their surfaces for reflecting light, and probably on the refraction and extinction of the solar rays in the atmosphere of Jupiter. By comparing the duration of the eclipses of all the satellites, an estimate may be formed of the influence of the causes enumerated. The variations in the distance of Jupiter and the sun from the earth, by changing the intensity of the light of the satellites, affects the apparent durations. The height of Jupiter above the horizon, the clearness of the air, and the power of the telescope employed in the observations, likewise affect their apparent duration; whence it not infrequently² happens that two observations of the same eclipse of the

first satellite differ by half a minute: for the second satellite the error may be more than double; for the third, the difference may exceed 3', and even 4' in the fourth satellite. When the immersion and emersion are both observed, the mean is taken, but an error of some seconds may arise, for the phase nearest the disc of Jupiter is liable to the greatest uncertainty on account of the light of the planet; so that an eclipse may be computed with more certainty than it can be observed. Although the eclipses of Jupiter's satellites may not be the most accurate method of finding the longitude, it is by much the easiest, as it is only requisite to reduce the time of the observation into mean time, and compare it with the time of the same eclipse computed for Greenwich in the *Nautical Almanac*,³ the difference of time is the longitude of the place of observation. The frequency of the eclipses renders this method very useful. The first satellite is eclipsed every forty-two hours; eclipses of the second recur in about four days, those of the third every seven days, and those of the fourth once in seventeen days. The latter is often a long time without being eclipsed, on account of the inclination of its orbit. Of course, the satellites are invisible all the time of Jupiter's immersion in the sun's rays.

925. Let mn , fig. 113, be the orbit of the satellite projected on the plane of Jupiter's orbit, then Jn will be the curtate distance of the satellite at the instant of conjunction, and mm' the projection of the arc described by the satellite on its orbit in passing through the shadow. In order to know the whole circumstances of an eclipse, the form and length of the shadow must first be determined; then its breadth where it is traversed by the satellite, which must be resolved into the polar co-ordinates of the motion of the satellite; whence may be found the duration of the eclipse, its beginning and its end. These are functions of the actual path of the satellite through the shadow, and of its projection mm' . If Jupiter were a sphere, the shadow would be a cone, with a circular base tangent to his surface; but as he is a spheroid, the cone has an elliptical base; its shape and size may be perfectly ascertained by computation, since both the form and magnitude of Jupiter are known.

926. The whole theory of eclipses may be analytically determined, if, instead of supposing the cone of the shadow to be traced by the revolution of the tangent AV , we imagine it to be formed by the successive intersections of an infinite number of plane surfaces, all of which touch the surfaces of the sun, and Jupiter in straight lines AaV .

927. A plane tangent to a curved surface not only touches the surface in one point, but it coincides with it through an indefinitely small space; therefore the co-ordinates of that point must not only have the same value in the finite equations of the two surfaces, but also the first differentials of these co-ordinates must be the same in each equation. Let the origin of the co-ordinates be in the centre of the sun; then if his mass be assumed to be a sphere of which R' is the radius, the equation of his surface will be

$$x'^2 + y'^2 + z'^2 = R'^2.$$

The general equation of a plane is

$$x = ay + bz + c,$$

a and b being the tangents of the angles this plane makes with the co-ordinate planes. In the point of tangency, $x, y, z,$ must not only be the same with $x', y', z',$ but $dx, dy, dz,$ must coincide with $dx', dy', dz';$ hence the equation of the plane and its differential become

$$\begin{aligned}x' &= ay' + bz' + c \\dx' &= ady' + bdz' .\end{aligned}$$

If this value of dx' be put in

$$x'dx' + y'dy' + z'dz' = 0,$$

which is the differential equation of the surface of the sun, it becomes

$$ax'dy' + bx'dz' + y'dy' + z'dz' = 0,$$

whatever the values of dy' and dz' may be. But this equation can only be zero under every circumstance when

$$\begin{aligned}ax' + y' &= 0 \\bx' + z' &= 0 .\end{aligned}$$

Thus the plane in question will touch the surface of the sun in a point A, when the following relations exist among the co-ordinates.

$$\begin{aligned}x'^2 + y'^2 + z'^2 &= R^2 \\ax' + y' = 0, \quad bx' + z' &= 0 \\x' &= ay' + bz' + c.\end{aligned} \tag{325}$$

928. This plane only touches the surface of the sun, but it must also touch the surface of Jupiter, therefore the same relations must exist between the co-ordinates of the surface of Jupiter and those of the plane, as exist between the co-ordinates of the plane, and those of the surface of the sun. So the equations must be similar in both cases. Without sensible error it may be assumed that Jupiter's equator coincides with his orbit. Were he a sphere, there would be no error at all, consequently it can only be of the order of his ellipticity into the inclination of his equator on his orbit, which is $3^\circ 5' 27''$.

The centre of the sun being the origin of the co-ordinates, if $SJ,$ the radius vector of Jupiter, be represented by $D,$ the equation of Jupiter's surface, considered as a spheroid of revolution, will be

$$(x_j - D)^2 + y_j^2 + (1 + r)^2 (z_j^2 - R_j^2) = 0, \tag{326}$$

R_j being half his polar axis, and r his ellipticity. The equations of contact are, therefore,

$$y_j + a(x_j - D) = 0$$

$$(1 + r)^2 z_j + b(x_j - D) = 0 \quad (327)$$

$$x_j - D = ay_j + bz_j + c - D.$$

929. These eight equations determine the line AaV , according to which the plane touches the sun and Jupiter; but in order to form the cone of the shadow, a succession of such plane surfaces must touch both bodies. The equations

$$x = ay + bz + c, \text{ and } dx = ady + bdz,$$

both belong to the same plane, but because one plane surface only differs from another by position, which depends on the tangents a and b , and on c , the distance from the origin of the coordinates; these quantities being constant for any one plane, it is evident they must vary in passing to that which is adjacent, therefore

$$dx = ady + bdz + yda + zdb + dc;$$

and subtracting

$$dx = ady + bdz,$$

there results

$$0 = y + z \frac{db}{da} + \frac{dc}{da},$$

in which b and c are considered to be functions of a .

If values of b , c , $\frac{db}{da}$, $\frac{dc}{da}$, be determined from (325), (327), and substituted in this equation, and in that of the plane, they will only contain a , the elimination of which will give the equation of the shadow; hence, if to these be added

$$x = ay + bz + c \quad (328)$$

$$0 = y + z \frac{db}{da} + \frac{dc}{da} \quad (329)$$

they will determine the whole theory of eclipses. If the bodies be spheres, it is only necessary to make $r = 0$.

930. In order to determine the equation of the shadow, values of

$$b, c, \frac{db}{da}, \frac{dc}{da},$$

must be found. The three first of equations (325) give

$$x'^2 (1 + a^2 + b^2) = R'^2,$$

and the three last give

$$x'(1+a^2+b^2)=c;$$

whence

$$c=R'\sqrt{1+a^2+b^2},$$

and

$$c-D=R'\sqrt{1+a^2+b^2}-D;$$

but from equations (326) and (327)

$$c-D=(1-r)R\sqrt{1+a^2+b^2}\frac{1}{(1+r)^2}$$

the square of r being neglected.

If

$$\frac{(1+r)R}{R'}=I,$$

[then]

$$f^2=\frac{D^2}{R'^2(1-I)^2}-1, \tag{330}$$

it may easily be found that

$$b=\left(1-\frac{Ir}{1-I}\right)\sqrt{f^2-a^2}$$

$$c=\frac{D}{1-I}-Ir\cdot\frac{R'^2}{D}(f^2-a^2);$$

whence

$$\frac{db}{da}=-\left(1-\frac{Ir}{1-I}\right)\frac{a}{\sqrt{f^2-a^2}}; \quad \frac{dc}{da}=Ir\frac{R'^2}{D}\cdot 2a;$$

and the equation

$$0=y+z\frac{db}{da}+\frac{dc}{da}$$

becomes

$$0=y-\left(1-\frac{Ir}{1-I}\right)\frac{az}{\sqrt{f^2-a^2}}+Ir\frac{R'^2}{D}\cdot 2a.$$

In order to have the equation of the shadow, a value of a must be found from this equation; which, with b and c , must be put in equation (328) of the plane. This will be accomplished with most ease by making $r=0$ in the preceding expression; whence

$$a = \frac{fy}{\sqrt{y^2 + z^2}}$$

is the value of a in the spherical hypothesis; but as Jupiter is a spheroid,

$$a = \frac{fy}{\sqrt{y^2 + z^2}} + qr;$$

consequently,

$$b = \left(1 - \frac{lr}{1-I}\right) \sqrt{f^2 - a^2} = \frac{fz}{\sqrt{y^2 + z^2}} - \frac{qry}{z} - \frac{lfrz}{(1-I)\sqrt{y^2 + z^2}}.$$

If this expression, together with the last value of a , and that of c be put in equation (328), it becomes

$$x = f\sqrt{y^2 + z^2} - \frac{lfrz^2}{(1-I)\sqrt{y^2 + z^2}} + \frac{D}{1-I} - \frac{lR'^2 \cdot f^2 z^2}{D(y^2 + z^2)};$$

whence⁵

$$\left(x - \frac{D}{1-I}\right)^2 = f^2(y^2 + z^2) - \frac{2f^2 \cdot lRz^2}{1-I} - \frac{2f^3 \cdot lr \cdot R'^2 \cdot z^2}{D\sqrt{y^2 + z^2}}.$$

931. At the summit of the cone y and z are zero, hence

$$x = \frac{D}{1-I} = SV, \text{ fig. 113,}$$

but for every other value of y and z , x is less than $\frac{D}{1-I}$, consequently the square root of f^2 in (330) must have a negative sign; and as D is very much greater than R' , $R'^2(1-I)^2$ may be neglected in comparison of D^2 , hence equation (330) becomes

$$f = \frac{-D}{R'(1-I)}, \text{ nearly;}$$

therefore the equation of the shadow of Jupiter is

$$\frac{R'^2(1-I)^2}{D^2} \left(\frac{D}{1-I} - x\right)^2 = y^2 + z^2 + \frac{2I}{1-I} \cdot rz^2 \left\{ \frac{R'}{\sqrt{y^2 + z^2}} - 1 \right\} \quad (331)$$

and that of the penumbra is

$$\frac{R^2(1+I)^2}{D^2} \left(x - \frac{D}{1+I} \right)^2 = y^2 + z^2 + \frac{2I}{1+I} \cdot r z^2 \left\{ \frac{R}{\sqrt{y^2 + z^2}} + 1 \right\} \quad (332)$$

932. In order to know the breadth of the shadow through which the satellite passes, and thence to compute the duration of the eclipse, it is necessary to determine the section made by a plane perpendicular to SV , fig. 113, the axis of the cone, and at the distance r from Jupiter. In this case

$$x = Sn = D + r,$$

and the equation of the shadow is

$$\frac{R'^2}{D^2} \{ DI - r(1-I) \}^2 = y^2 + z^2 + \frac{2I}{1-I} \cdot r z^2 \left\{ \frac{R'}{\sqrt{y^2 + z^2}} - 1 \right\}.$$

If at first $r = 0$,

$$\sqrt{y^2 + z^2} = RI \left\{ 1 - \frac{r(1-I)}{DI} \right\}.$$

If this be put in the term which has r as a factor, and if to abridge

$$r' = \frac{r \left(1 + \frac{r}{D} \right)}{1 - \frac{r(1-I)}{DI}},$$

the result will be

$$(1+r)^2 R'^2 \left\{ 1 - \frac{r(1-I)}{DI} \right\}^2 = y^2 + z^2 + 2z^2 r',$$

the equation to an ellipse whose eccentricity is r' , and half the greater axis,

$$= (1+r) R' \left\{ 1 - \frac{r(1-I)}{DI} \right\} = a \quad (333)$$

$(1+r) R'$ is the equatorial radius of Jupiter; hence the section of Jupiter's shadow at the distance of the satellite is

$$a^2 - y^2 = (1+r')^2 z^2,$$

and $2\mathbf{a}$ is its greatest breadth. $2\mathbf{a}$ is the actual path of the satellite through the shadow, and mm' , fig. 113, is its projection on the orbit of Jupiter.

If \mathbf{l} be made negative in the values of \mathbf{a} and \mathbf{r}' , the preceding equation will be the section of the penumbra at the distance r from the centre of Jupiter, the difference of the two sections

$$\frac{2r}{D\mathbf{l}}(1 + \mathbf{r}')R' = \frac{2R'r}{D} \text{ nearly,}$$

is the greatest breadth of the penumbra at that point, R' being the semidiameter of the sun.

933. To express the section of the shadow in polar co-ordinates of the motion of the satellite, let z be the height of a satellite above the orbit of Jupiter at the instant of its conjunction, r its radius vector, the projection of which on the orbit of Jupiter is $Jn = \sqrt{r^2 - z^2}$, fig. 113. Let v' be the angle described by the satellite from the instant of conjunction by its synodic motion, and projected on Jupiter's orbit, of which $\pm mn$ is the corresponding arc; and let SV be the axis of the co-ordinates x , then

$$y^2 = (r^2 - z^2) \sin^2 v'$$

which makes the equation of the section of the surface of the shadow

$$(r^2 - z^2) \sin^2 v' = \mathbf{a}^2 - (1 + \mathbf{r}') z^2,$$

or rejecting quantities of the order z^4 , $z^2 \sin^2 v'$,

$$r^2 \sin^2 v' = \mathbf{a}^2 - (1 + \mathbf{r}')^2 z^2.$$

But as r is nearly constant, we have

$$z = r \left\{ s + \sin v' \cdot \frac{ds}{dv'} + \frac{1}{2} \sin^2 v' \cdot \frac{d^2s}{dv'^2} + \&c. \right\}, \quad (334)$$

s being the tangent of the latitude of the satellite above the orbit of Jupiter at n at the instant of conjunction

$$z^2 = r^2 \left\{ s^2 + 2 \sin v' \cdot \frac{sds}{dv'} \right\} \text{ nearly,}$$

hence

$$r^2 \sin^2 v' = \mathbf{a}^2 - (1 + \mathbf{r}')^2 r^2 s^2 - 2r^2 (1 + \mathbf{r}')^2 \frac{sds}{dv'} \sin v'$$

from which⁶

$$\sin v' = -(1 + r')^2 \cdot \frac{sds}{dv'} \pm \sqrt{\left\{ \frac{a^2}{r^2} - (1 + r')^2 s^2 \right\}}.$$

With the positive sign of the radical this formula is the sine of the arc nm' described by the satellite in its synodic motion from conjunction to emersion on the orbit of Jupiter, and with the negative sign it is the arc mn from immersion to conjunction.

934. In order to find the duration of the eclipse, let T be the time employed by the satellite to describe a , half the breadth of the shadow on its orbit by its synodic motion, and let t be the time it takes to describe its projection v' . Then nt and Mt being the mean motions of the satellite and Jupiter, it is evident that dv' the arc described by the satellite during the time dt , must be equal to the difference of the mean motions of the satellite and Jupiter, or $dv' = dt(n - M)$, if the disturbing forces be omitted; but if w be the indefinitely small change in the equation of the centre during the time dt , then

$$dv' = dt(n - M)\{1 + w\},$$

or

$$\frac{dv'}{(n - M)dt} = 1 + w.$$

Again, since a has been taken to represent the mean distance of the satellite m from Jupiter, $\frac{a}{a}$ is the sine of the angle under which a , half the breadth of the shadow, is seen from the centre of Jupiter. Let x be this angle, which is very small, and may be taken for its sine, then

$$t = \frac{Tv'(1 - w)}{x}.$$

But v' is so small that

$$t = \frac{T(1 - w)\sin v'}{x};$$

and if the preceding values of $\sin v'$ be substituted, putting also ax for a , the result will be⁷

$$t = T(1 - w) \left\{ -(1 + r')^2 \cdot \frac{sds}{x dv'} \pm \sqrt{\left\{ \frac{a^2}{r^2} - (1 + r')^2 \frac{s^2}{x^2} \right\}} \right\}.$$

If all the inequalities be omitted, except the equations of the centre,

$$r = a \left(1 - \frac{1}{2} w \right);$$

and as the same equation exists, even including the principal inequalities⁸

$$t = T(1-w) \left\{ -(1+r')^2 \cdot \frac{s}{x} \cdot \frac{ds}{dv'} \pm \sqrt{\left[1 + \frac{1}{2}w + (1+r') \frac{s}{x} \right] \left[1 + \frac{1}{2}w - (1+r') \frac{s}{x} \right]} \right\} \quad (335)$$

and if t' be the whole duration of the eclipse,

$$t' = 2T(1-w) \sqrt{\left[1 + \frac{1}{2}w + (1+r') \frac{s}{x} \right] \left[1 + \frac{1}{2}w - (1+r') \frac{s}{x} \right]}$$

whence may be derived

$$s = \frac{x \sqrt{4T^2(1-w) - t'^2}}{2T(1+r')(1-w)}.$$

Since s is given by the equations of latitude, this expression will serve for the determination of the arbitrary constant quantities that it contains, by choosing those observations of the eclipses on which the constant quantities have the greatest influence.

935. Both Jupiter and the satellite have been assumed to move in circular orbits, but \mathbf{a} , half the breadth of the shadow, varies with their radii vectores. D' being the mean distance of Jupiter from the sun, $D' - \mathbf{d}D$ may represent the true distance, so that equation (333) becomes

$$(1+r)R' \left\{ \frac{1}{2}w - \frac{\mathbf{d}D}{D'} \right\} \frac{(1-I)a}{ID'}$$

$\frac{1}{2}w$ is always much less than

$$\frac{\mathbf{d}D}{D'} = H \cos(Mt + E - \Pi) = H \cos V,$$

so the change in \mathbf{a} is

$$-\mathbf{a} \frac{(1-I)}{I} \cdot \frac{a}{D'} H \cos V;$$

and the value of $\frac{\mathbf{a}}{x}$ becomes

$$\frac{\mathbf{a}}{x} \left\{ 1 - \frac{(1-I)}{I} \cdot \frac{a}{D'} H \cos V \right\}.$$

In this function x is relative to the mean motions and mean distances of the satellite from Jupiter, and of Jupiter from the sun.

936. Since the breadth of the shadow is diminished by this cause, the time T of describing half of it will be diminished by

$$T \frac{(1-I)}{I} \frac{a}{D'} H \cos V;$$

but as the synodic motion in the time dt is nearly

$$(n-M) dt \left\{ 1 + w - \frac{2M}{n-M} H \cos V \right\},$$

the time will be increased by

$$T \left\{ \frac{2M}{n-M} H \cos V - w \right\}.$$

Omitting w , the time T on the whole will become from these two causes

$$T \left\{ 1 + \left(\frac{2M}{n-M} - \frac{1-I}{I} \cdot \frac{a}{D'} \right) H \cos V \right\}; \quad (336)$$

but this is only sensible in the fourth satellite.

937. The arcs v_j and \mathbf{x} are so small, that no sensible error arises from taking them for their sine, and the contrary; indeed, the observations of the eclipses are liable to so many sources of error, that theory will determine these phenomena with most precision, notwithstanding these approximate values; should it be necessary, it is easy to include another term of the series in article 933.

938. The duration of the eclipses of each satellite may be determined from equation (335).

Delambre⁹ found, from the mean of a vast number of observations, that half the mean duration of the eclipses of the fourth satellite in its nodes, is $T = 3204''.4$, which is the maximum; $\mathbf{x} = 7650''.6$ is the mean synodic motion of the satellite during the time T . In article 893, $\mathbf{r} = 0.0713008$. The semidiameter of Jupiter is by observation, $2(1+\mathbf{r})R_j = 39''$. R_j is the semidiameter of the sun seen from Jupiter. The semidiameter of the sun, at the mean distance of the earth, is $1923''.26$; it is therefore $\frac{1923''.26}{D'}$, when seen from Jupiter; $D' = 5.20116636$, is the mean distance of Jupiter from the sun, and as $a_3 = 25.4359$, it is easy to find that¹⁰

$$\mathbf{r}' = \frac{\mathbf{r} \left(1 + \frac{a_3}{D'} \right)}{1 - \frac{(1-I)}{I} \cdot \frac{a_3}{D'}}$$

becomes $r' = 0.0729603$. $w = \frac{dv_3}{n_3 dt}$ is the indefinitely small variation in the equation of the equation of the centre during the time dt ; and if the greatest term alone be taken,

$$w = 0.0145543 \cos(n_3 t + \epsilon_3 - \mathbf{v}_3);$$

but the time T must be multiplied by

$$1 + \left\{ \frac{2M}{n_3 - M} - \frac{(1-I)}{I} \cdot \frac{a_3}{D'} \right\} H \cos V,$$

H being the eccentricity of Jupiter's orbit; as the numerical values of all the quantities in this expression are given, this factor is $1 - 0.0006101 \cos V$; and if

$$z_3 = \frac{(1+r')s_3}{\mathbf{x}},$$

s_3 being the latitude of the fourth satellite, given in (324); then

$$\begin{aligned} z_3 = & +1.352380 \sin(v_3 + 46^\circ.241 - 49''.8t) \\ & -0.125759 \sin(v_3 + 74^\circ.969 + 2439''.07t) \\ & +0.020399 \sin(v_3 + 187^\circ.4931 + 9143''.6t) \\ & +0.000218 \sin(v_3 + 273^\circ.2889 + 43323''.9t). \end{aligned}$$

If the square of w be omitted, it reduces the quantity under the radical in equation (327) to $1 + w - z^2$; and if the products of w and H by $\frac{z_3 dz_3}{dv_3}$ be neglected, the expression (335) becomes

$$t = -118''.9 \frac{z_3 dz_3}{dv_3} \pm 3204''.4 (1 - w - 0.0006101 \sin V) \sqrt{1 + w - z_3^2}.$$

From this expression it is easy to find the instants of immersion and emersion; for t was shown to be the time elapsed from the instant of the conjunction of the satellite projected on the orbit of Jupiter in n , which instant may be determined by the tables of Jupiter, and the expressions in (323) and (324) of v_3 and s_3 , the longitude and latitude of the satellite.

The whole duration of the eclipses of the fourth satellite will be

$$6408''.7 (1 - w - 0.0006101 \sin V) \cdot \sqrt{1 + w - z_3^2}.$$

939. With regard to the eclipses of the third satellite, $T = 2403''.8$, which is the maximum. The mean motion of the satellite, during the time T , is

$$\mathbf{x} = 13416''.8, \quad a_2 = 14.461893;$$

whence

$$\mathbf{r}' = 0.072236;$$

and if only the three greatest terms of v_2 , in equation (321) be employed, $w = \frac{dv_2}{n_2 dt}$ becomes

$$\begin{aligned} w = & +0.00268457 \cos(n_2 t + \epsilon_2 - \mathbf{v}_2) \\ & +0.00118848 \cos(n_2 t + \epsilon_2 - \mathbf{v}_3) \\ & -0.00126952 \cos(n_1 t - n_2 t + \epsilon_1 - \epsilon_2). \end{aligned}$$

The factor in (336) becomes, with regard to this satellite,

$$-0.00039871 \sin V.$$

Then, if $\mathbf{z}_2 = \frac{(1 + \mathbf{r}') s_2}{\mathbf{x}}$, s_2 being the latitude of the third satellite,¹¹

$$\begin{aligned} \mathbf{z}_2 = & +0.864850 \sin(v_2 + 46^\circ.241 - 49''.8t) \\ & -0.059101 \sin(v_2 + 187^\circ.4931 + 9143''.6t) \\ & -0.008961 \sin(v_2 + 74^\circ.969 + 2439''.08t) \\ & +0.004570 \sin(v_2 + 273^\circ.2889 + 43323''.9t). \end{aligned}$$

Hence

$$t = -167''.64 \cdot \frac{\mathbf{z}_2 d\mathbf{z}_2}{dv_2} \pm 2403''.8 (1 - w - 0.00039871 \sin V) \sqrt{1 + w - \mathbf{z}_2^2};$$

from whence the instants of immersion and emersion may be computed, by help of the tables of Jupiter, and of the longitude and latitude of the third satellite in (321) and (322).

The whole duration of the eclipses of the third satellite is

$$4807''.5 (1 - w - 0.00039871 \sin V) \sqrt{1 + w - \mathbf{z}_2^2}.$$

940. The value of T from the eclipses of the second satellite, is $T = 1936''.13$; and \mathbf{x} , the synodic mean motion of the second satellite during the time T , is $\mathbf{x} = 21790''.4$; $a_1 = 9.066548$, $\mathbf{r}' = 0.0718862$. If we only take the greatest terms of v_1 in (319)

$$w = \frac{dv_j}{ndt} \text{ will be}$$

$$w = +0.00057797 \cos(n_1 t + \epsilon_1 - \nu_2) \\ + 0.0187249 \cos 2(n_1 t - n_2 t + \epsilon_1 - \epsilon_2).$$

The factor (336) has no sensible effect on the eclipses, either of this satellite or the first, and may therefore be omitted.

If $z_j = \frac{(1+r')S_j}{x}$, s_j being the latitude of the second satellite in (320); then

$$z_j = +0.507629 \sin(\nu_j + 46^\circ.241 - 49''.8t) \\ - 0.076569 \sin(\nu_j + 273^\circ.2889 - 43323''.9t) \\ - 0.005571 \sin(\nu_j + 187^\circ.4931 - 9143''.6t) \\ - 0.0009214 \sin(\nu_j + 75^\circ.059 - 2439''.07t)$$

[and]¹²

$$t = -204''.54 \frac{z_j dz_j}{dv_j} \pm 1936''.13 (1-w) \sqrt{1+w-z_j^2}$$

and the whole duration of the eclipses of the second satellite is

$$3872''.25 (1-w) \sqrt{1+w-z_j^2}.$$

941. The value of T from the eclipses of the first satellite, is $T = 1527''$, and the mean synodic motion of the first satellite during the time T , is $x = 34511''.2$; and as $a = 5.698491$, $r' = 0.0716667$. If only the greatest term of v in (318) be taken

$$w = \frac{dv}{ndt} \text{ becomes}$$

$$w = 0.0079334 \cos 2(nt - n_1 t + \epsilon - \epsilon_1);$$

and if $z = \frac{(1+r')s}{x}$, s being the latitude of the first satellite in article 908, then

$$z = +0.345364 \sin(\nu + 46^\circ.241 - 49''.8t) \\ - 0.001057 \sin(\nu + 273^\circ.2889 + 43323''.9t) \\ - 0.000256 \sin(\nu + 187^\circ.4931 + 9143''.6t);$$

also

$$t = -255''.49 \frac{z dz}{dv} \pm 1527''(1-w) \sqrt{1+w-z^2},$$

and the whole duration of the eclipses of the first satellite is

$$3054''(1-w) \sqrt{1+w-z^2}.$$

942. The errors to which the durations of the eclipses are liable, may be ascertained. Equation (333) divided by a , or which is the same thing $\frac{a}{a}$ is the sine of the angle described by each satellite during half the duration of its eclipses, supposing the satellite to be eclipsed the instant it enters the shadow. This angle, divided by the circumference, and multiplied by the time of a synodic revolution of the satellite, will give half the duration of the eclipse; and, comparing it with the observed semi-duration, the errors, arising from whatever cause, will be obtained. If q, q_1, q_2, q_3 , be this angle for each satellite, equation (333) gives

$$\begin{aligned} \frac{(1+r)R_v}{a_3} \left\{ \frac{a_3}{a} - \frac{(1-I)}{I} \cdot \frac{a_3}{D'} \right\} &= \sin q \\ \frac{(1+r)R_v}{a_3} \left\{ \frac{a_3}{a_1} - \frac{(1-I)}{I} \cdot \frac{a_3}{D'} \right\} &= \sin q_1 \\ \frac{(1+r)R_v}{a_3} \left\{ \frac{a_3}{a_2} - \frac{(1-I)}{I} \cdot \frac{a_3}{D'} \right\} &= \sin q_2 \\ \frac{(1+r)R_v}{a_3} \left\{ 1 - \frac{(1-I)}{I} \cdot \frac{a_3}{D'} \right\} &= \sin q_3. \end{aligned}$$

By what precedes, $I = 0.105469$,

$$\frac{(1+r)R_v}{D'} = 0.000094549;$$

whence

$$\begin{aligned} \frac{1}{a} - 0.000801823 &= \sin q \\ \frac{1}{a_1} - 0.000801823 &= \sin q_1 \\ \frac{1}{a_2} - 0.000801823 &= \sin q_2 \\ \frac{1}{a_3} - 0.000801823 &= \sin q_3; \end{aligned}$$

and if the values of a_1, a_2, a_3 , in article 87, be substituted,

$$\begin{aligned} q &= 10^{\circ}.0602 \\ q_1 &= 6^{\circ}.2861 \\ q_2 &= 3^{\circ}.919 \\ q_3 &= 2^{\circ}.2072. \end{aligned}$$

These are the angles described by the satellites during half the eclipse; and when divided by the circumference, and multiplied by the time of the synodic revolution of the satellites, they will give the duration of half the eclipse, whence half the duration of the eclipses are

1 st Satellite	1602".46
2 nd Satellite	2010".72
3 rd Satellite	2527".62
4 th Satellite	3328".01.

The semi-durations, from observation, are,

1 st Satellite	1527"
2 nd Satellite	1936"
3 rd Satellite	2404"
4 th Satellite	3204".

943. The observed values are less than the computed, for they are diminished by the whole of the time that the discs of the satellites take to disappear after their centres have entered the shadow. The duration may be lessened by the refraction of the solar light on Jupiter's atmosphere, but it is augmented by the penumbra. These two last causes however are not sufficient to account for the difference between the computed and the observed semi-durations; therefore the time that half the discs of the satellites employ to pass into the shadow must be computed.

944. The effects of the penumbra, and of the reflected light of the sun on the atmosphere of Jupiter, are inconsiderable with regard to the first satellite. In order to have the breadth of the disc of the first satellite seen from Jupiter, let the density of this satellite be the same with that of Jupiter, and the mass and semidiameter of the planet be unity; then the apparent semidiameter of the satellite seen from the centre of Jupiter, is $\frac{\sqrt[3]{m}}{a}$; and substituting the values of a and m ,

$$\frac{\sqrt[3]{m}}{a} = 15' 10".42.$$

This angle multiplied by $1^{\text{day}}.769138$, and divided by 360° , gives $41".44$ for the time half the disc would take to pass into the shadow. Subtracting it from $1602".46$, the remainder

1561".02 is the computed semi-duration, which is greater than the observed time; and yet there is reason to believe that the satellite disappears before it is quite immersed. It appears then, that the diameter of Jupiter must be diminished by at least a 50th part, which reduces it from 39" to 38". The most recent observations give 38".44 for the apparent equatorial diameter of Jupiter, and 35".65 for his polar diameter.

By this method it is computed that the discs of the satellites, seen from the centre of Jupiter, and the time they take to penetrate perpendicularly into the shadow, are

	Discs	Times
1 st Satellite	1820".83	82".888
2 nd Satellite	1298".37	115".362
3 rd Satellite	1271".19	227".744
4 th Satellite	566".7	237".352.

Whence the times of immersion and emersion of the satellites and of their shadows on the disc of Jupiter may be found, when they pass between him and the sun.

945. The observations of the eclipses of Jupiter by his satellites, may throw much light on their theory. The beginning and end of their transits may almost always be observed, which with the passage of the shadow afford four observations; whereas the ellipse of a satellite only gives two. Laplace¹³ thinks these phenomena particularly worthy of the attention of practical astronomers.

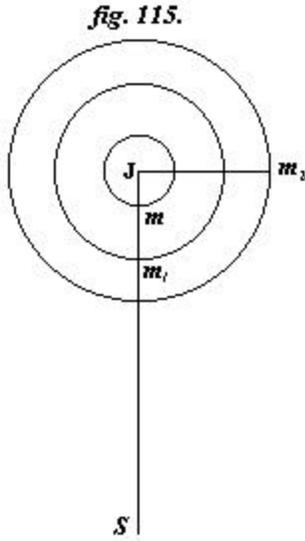
946. In the preceding investigations, the densities of the satellites were assumed to be the same with that of Jupiter. By comparing the computed times with the observed times of duration, the densities of the satellites will be found when their masses shall be accurately ascertained.

947. The perturbations of the three first satellites have a great influence on the times of their eclipses. The principal inequality of the first satellite retards, or advances its eclipses 72.41 seconds at its maximum. The principal inequality of the second satellite accelerates or retards its eclipses by 343".2, at its maximum, and the principal inequality of the third satellite advances or retards its eclipses by 261".9 at its maximum.

948. Since the perturbations of the satellites depend only on the differences of their mean longitudes, it makes no alteration in the value of these differences, whether the first point of Aries be assumed as the origin of the angles, or *SJ* the radius vector of Jupiter supposed to move uniformly round the sun. If the angles be estimated from *SJ*, nt , n_1t , n_2t , become the mean synodic motion of the three first satellites; and in both cases

$$nt - 3n_1t + 2n_2t + \epsilon - 3\epsilon_1 + 3\epsilon_2 = 180.$$

Suppose the longitudes of the epochs of the two first satellites to be zero or $\epsilon = 0$, $\epsilon_1 = 0$, so that these two bodies are in conjunction with Jupiter when $t = 0$, then it follows that $\epsilon_2 = 90^\circ$, and thus when the two first satellites are in conjunction, the third is a right angle in advance, as in fig. 115; and the principal inequalities of the three first satellites become



$$d v = +1636'' .4 \sin (2 n t - 2 n_1 t)$$

$$d v_1 = -3862'' .3 \sin (n t - n_1 t)$$

$$d v_2 = + 261'' .86 \sin (n_1 t - n_2 t).$$

In the eclipses of the first satellite at the instant of conjunction $n t = 0$, or it is equal to a multiple of 360 . Let

$$2 n - 2 n_1 = n + w, \text{ or } n - 2 n_1 = w$$

then

$$d v = 1636'' .4 \sin w t.$$

In the eclipses of the second satellite at the instant of conjunction $n_1 t = 0$, or it is equal to a multiple of 360° ; hence

$$d v_1 = -3862'' .3 \sin w t.$$

Lastly, in the eclipses of the third satellite, $n_2 t + \epsilon_2 = 0$, or it is a multiple of 360° at the instant of conjunction, hence¹⁴

$$d v_2 = 261'' .86 \sin w t.$$

Thus it appears that the periods of these inequalities in the eclipses are the same, since they depend on the same angle. This period is equal to the product of $\frac{n}{n - 2 n_1}$ by the duration of the synodic revolution of the first satellite, or to 437.659 days, which is perfectly conformable to observation.

949. On account of the ratio

$$n t - 3 n_1 t + 2 n_2 t + \epsilon - 3 \epsilon_1 + 3 \epsilon_2 = 180^\circ,$$

the three first satellites never can be eclipsed at once, neither can they be seen at once from Jupiter when in opposition or conjunction; for if

$$n t + \epsilon, n_1 t + \epsilon_1, n_2 t + \epsilon_2,$$

be the mean synodic longitudes, in the simultaneous eclipses of the first and second

$$n t + \epsilon = n_1 t + \epsilon_1 = 180^\circ;$$

and from the law existing among the mean longitudes, it appears that

$$n_2t + \epsilon_2 = 270.$$

In the simultaneous eclipses of the first and third satellites

$$nt + \epsilon = n_2t + \epsilon_2 = 180^\circ,$$

and on account of the preceding law, $n_1t + \epsilon_1 = 120$.

Lastly in the simultaneous eclipses of the second and third satellites

$$n_1t + \epsilon_1 = n_2t + \epsilon_2 = 180^\circ;$$

hence $nt + \epsilon = 0$, thus the first satellite in place of being eclipsed, may eclipse Jupiter.

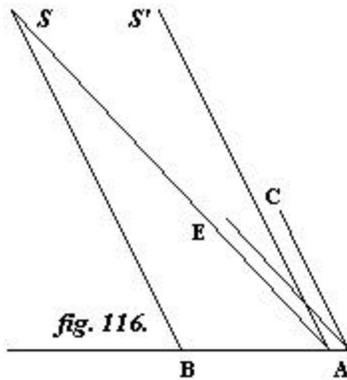
Thus in the simultaneous eclipses of the second and third satellites, the first will always be in conjunction with Jupiter; it will always be in opposition in the simultaneous transits of the other two.

950. The comparative distances of the sun and Jupiter from the earth may be determined with tolerable accuracy from the eclipses of the satellites. In the middle of an eclipse, the sidereal position of the satellite, and the centre of Jupiter is the same when viewed from the centre of the sun, and may easily be computed from the tables of Jupiter. Direct observation, or the known motion of the sun gives the position of the earth as seen from the centre of the sun; hence, in the triangle formed by the sun, the earth, and Jupiter, the angle at the sun will be known; direct observation will give that at the earth, and thus at the instant of the middle of the eclipse, the relative distances of Jupiter from the earth and from the sun, may be computed in parts of the distance of the sun from the earth. By this method, it is found that Jupiter is at least five times as far from us as the sun is when his apparent diameter is $36''.742$. The diameter of the earth at the same distance, would only appear under an angle of $3''.37$. The volume of Jupiter is therefore at least a thousand times greater than that of the earth.

951. On account of Jupiter's distance, some minutes elapse from the instant at which an eclipse of a satellite begins or ends, before it is visible at the earth.

Roëmer observed, that the eclipses of the first satellite happened sooner, than they ought by computation when Jupiter was in opposition, and therefore nearer the earth; and later when Jupiter was in conjunction, and therefore farther from the earth. In 1675, he shewed¹⁵ that this circumstance was owing to the time the light of the satellite employed in coming to the observer at the different distances of Jupiter. It was objected to this explanation, that the circumstance was not indicated by the eclipses of the other satellites, in which it was difficult to detect so small a quantity among their numerous inequalities then little known; but it was afterwards proved by Bradley's discovery of the aberration of light in the year 1725;¹⁶ when he was endeavouring to determine the parallax of β Draconis. He observed that the stars had a small annual motion. A star near the pole of the ecliptic appears to describe a small circle about it parallel to the ecliptic, whose diameter is $4''$, the pole being the true place of the star. Stars situate in the ecliptic appear to describe arcs of the ecliptic of $40''$ in length, and all stars between these two positions

seem to describe ellipses whose greater axes are $40''$ in length, and are parallel to the ecliptic. The lesser axes vary as the sine of the star's latitude. This apparent motion of the stars arises



from the velocity of light combined with the motion of the earth in its orbit. The sun is so very distant, that his rays are deemed parallel; therefore let $S'A, SB$, fig. 116, be two rays of light coming from the sun to the earth moving in its orbit in the direction AB . If a telescope be held in the direction AC , the ray $S'A$ in place of going down the tube CA will impinge on its side, and be lost in consequence of the telescope being carried with the earth in the directions AB ; but if the tube be in a position SEA , so that $BA : BS$ as the velocity of the earth to the velocity of light; the ray will pass in the diagonal SA , which is the component of these two velocities, that is, it will pass through the axis of the telescope while carried parallel to itself with the earth. The star appears in the direction AS , when it really is in the direction AS' ; hence $S'AS = ASB$ is the quantity or angle of aberration, which is always in the direction towards which the earth is moving.

Delambre¹⁷ computed from 1000 eclipses of the first satellite, that light comes from the sun at his mean distance of about 95 millions of miles in $8' 13''$;¹⁸ therefore the velocity of light is more than ten thousand times greater than the velocity of the earth, which is nineteen miles in a second: hence BS is about 10,000 times greater than AB , consequently the angle ASB is very small. When EAB is a right angle, ASB is a maximum, and then

$$\sin ASB : 1 :: AB : BS :: \text{velocity of earth} : \text{velocity of light};$$

but $ASB =$ the aberration; hence the sine of the greatest aberration is equal to

$$\frac{\text{rad. velocity of light}}{\text{velocity of light}} = \sin 20''.25$$

by the observation of Bradley¹⁹ which perfectly correspond with the maximum of aberration computed by Delambre²⁰ from the mean of 6000 eclipses of the first satellite.

This coincidence shews the velocity of light to be uniform within the terrestrial orbit, since the one is derived from the velocity of light in the earth's orbit, and the other from the time it employs to traverse its diameter. Its velocity is also uniform in the space included in the orbit of Jupiter, for the variations of his radius vector are very sensible in the times of the eclipses of his satellites, and found to correspond exactly with the uniform motion of light.

If light be propagated in space by the vibrations of an elastic fluid, its velocity being uniform, the density of the fluid must be proportional to its elasticity.

952. The concurrent exertions of the most eminent practical and scientific astronomers have brought the theory of the satellites to such perfection, that calculation furnishes more accurate results than observation. Galileo²¹ obtained approximate values of the mean distances and periodic times of the satellites from their configurations, and Kepler²² was able to deduce from these imperfect data, proofs that the squares of their periodic times are proportional to the

cubes of their mean distances, establishing an analogy between these bodies and the planetary systems, subsequently confirmed.

Bradley²³ found that the two first satellites return to the same relative positions in 437 days. Wagentin discovered a similar inequality in the third of the same period, which was concluded to be the cycle of their disturbances.

In the year 1766, the Academy of Sciences at Paris proposed the theory of the satellites of Jupiter as a prize question, which produced a masterly solution of the problem by Lagrange.²⁴ In the first approximation he obtained the inequalities depending on the elongations previously discovered by Bradley; in the second, he obtained four equations of the centre for each satellite, and by the same analysis shewed that each satellite has four principle equations in latitude, which he represented by four planes moving on each other at different but constant inclinations; however, his equations of the latitude were incomplete, from the error of assuming Jupiter's equator to be on the plane of his orbit. It was reserved for Laplace²⁵ to perfect this important theory, by including in these equations the inclination of Jupiter's equator, the effects of his nutation, precession, and the displacement of his orbit, and also by the discovery of the four fixed planes, of the libration, and of the law in the mean longitudes, discoveries that rank high among the many elegant monuments of genius displayed in his system of the world. The perfect harmony of these laws with observation, affords one of the numerous proofs, of the universal influence of gravitation. They are independent of secular inequalities, and of the resistance of a rare medium in space, since such resistance would only cause secular inequalities so modified by the mutual attraction of the satellites, that the secular equation of the first, minus three times that of the second, plus twice that of the third, would always be zero; therefore the inequalities in the return of the eclipses, whose period is 437 days, will always be the same.

953. The libration by which the three first satellites balance each other in space, is analogous to a pendulum performing an oscillation in 1135 days. It influences all the secular variations of the satellites, although only perceptible at the present time in the inequality depending on the equation of the centre of Jupiter; and as the observations of Sir William Herschel²⁶ shew that the periods of the rotation of the satellites are identical with the times of their revolutions, the attraction of Jupiter affects both with the same secular inequalities.

954. Thus Jupiter's three first satellites constitute a system of bodies mutually connected by the inequalities and relations mentioned, which their reciprocal action will ever maintain if the shock of some foreign cause does not derange their motion and relative position: as, for instance, if a comet passing through the system, as that of 1770 appears to have done, should come in collision with one of its bodies. That such collisions have occurred since the origin of the planetary system, is probable: the shock of a comet, whose mass only equaled the one hundred thousandth part of that of the earth, would suffice to render the libration of the satellites sensible; but since all the pains bestowed by Delambre upon the subject did not enable him to detect this, it may be concluded that the masses of any comets which may have impinged upon one of the three satellite's nearest to Jupiter must have been extremely small, which corresponds with what we have already had occasion to observe on the tenuity of the masses of the comets, and their hitherto imperceptible influence on the motions of the solar system.

955. To complete the theory, thirty-one unknown quantities remained to be derived from observation, all of which Delambre determined from 6000 eclipses, and with these data he

computed tables of the motions of the satellites from Laplace's formulae, subsequently brought to great perfection by Mr. Bouvard.

The Satellites of Saturn

956. Saturn is surrounded by a ring, and seven satellites revolve from west to east round him, but their distance from the earth is so great that they are only discernible by the aid of very powerful telescopes, and consequently their eclipses have not been determined, their mean distances and periodic times alone have been ascertained with sufficient accuracy to prove that Kepler's third law extends to them. If $8''.1$ the apparent equatorial semidiameter of Saturn in his mean distance from the sun be assumed as unity, the mean distances and periodic times of the seven satellites are,

	Mean distance	Periodic times. days
1 st	3.351	0.94271
2 nd	4.300	1.37024
3 rd	5.284	1.8878
4 th	6.819	2.73948
5 th	9.524	4.51749
6 th	22.081	15.9453
7 th	64.359	79.3296

The masses of the satellites and rings and the compression of Saturn being unknown, their perturbations cannot be determined. The orbits of the six interior satellites remain nearly in the plane of Saturn's equator, owing to his compression, and the reciprocal attraction of the bodies.

The orbit of the seventh satellite has a motion nearly uniform on a fixed plane passing between the orbit and equator of that planet, inclined to that plane at an angle of $15^{\circ}.264$. The nodes have a retrograde annual motion of $304''.6$; the fixed plane maintains a constant inclination of $21^{\circ}.6$ to Saturn's equator, but the approximation must be imperfect that results from data so uncertain.

957. The action of Saturn on account of his compression, retains the rings and the orbits of the six first satellites in the plane of his equator. The action of the sun constantly tends to make them deviate from it; but as this action increases very rapidly, and nearly as the 5th power of the radius of the orbit of the satellite, it is sensible in the seventh only. This is also the reason why the orbits of Jupiter's satellites are more inclined in proportion to their greater distance from their primary, because the attraction of his equatorial matter decreases rapidly, while that of the sun increases.

When the seventh satellite is east of the planet, it is scarcely perceptible from the faintness of its light, which must rise from spots on the hemisphere presented to us. Now, in order to exhibit always the same appearance like the moon and satellites of Jupiter, it must revolve on its axis in a time equal to that in which it revolves round its primary. Thus the

equality of the time of rotation to that of revolution seems to be a general law in the motion of the satellites.

The compression of Saturn must be considerable, its revolution being performed in 11^h 42' 43", nearly the same with that of Jupiter.

Satellites of Uranus

958. The slow motion of Uranus in its orbit shows it to be on the confines of the solar system. Its distance is so vast that its apparent diameter is but 3".9, its satellites are therefore only within the scope of instruments of very high powers; Sir William Herschel discovered six revolving in circular orbits nearly perpendicular to the plane of the ecliptic. Taking the semidiameter of the planet for unity, their mean distances and periodic times are

	Mean distance	Periodic times. days
1 st	13.120	5.8926
2 nd	17.022	8.7068
3 rd	19.845	10.9611
4 th	22.752	13.4559
5 th	45.509	38.0750
6 th	91.008	107.6944

subject, therefore, to the third law of Kepler. The compression of their primary and their reciprocal attraction retains their orbits in the plane of the planet's equator.

Notes

¹ This chapter is numbered "IX" in the 1st edition.

² This reads "unfrequently" in the 1st edition.

³ See note 22, *Preliminary Dissertation*.

⁴ This reads "tangence" in the 1st edition.

⁵ The last two terms read $-\frac{2f \cdot {}^2I r z^2}{1-I} - \frac{2f \cdot {}^3I r \cdot R'^2 \cdot z^2}{D\sqrt{y^2+z^2}}$ in the 1st edition.

⁶ Punctuation added.

⁷ The closing parenthesis is omitted in the 1st edition.

⁸ There is a misplaced prime in $\frac{ds}{dv'}$ that reads $\frac{ds}{dv}$ in the 1st edition.

⁹ See note 54, *Preliminary Dissertation*.

¹⁰ The denominator reads $1 - \frac{(1-I)}{I} \cdot \frac{a^3}{D'}$ in the 1st edition.

¹¹ The argument in the 4th term reads $v_2 + 273.2889 + 43323''9t$ in the 1st edition.

¹² The last term reads $\pm 1936''.13(1-w)\sqrt{1+w-z^2}$ in the 1st edition.

¹³ See note 4, *Introduction*.

¹⁴ This reads $dv_2 = 26186\sin yt$ in the 1st edition.

¹⁵ *shewed*. Archaic use of “showed.”

¹⁶ See note 38, *Preliminary Dissertation*.

¹⁷ See note 9.

¹⁸ This reads $8'.13''$ in the 1st edition.

¹⁹ See note 17.

²⁰ See note 9.

²¹ See note 1, *Introduction*.

²² See note 3, *Preliminary Dissertation*.

²³ See note 9.

²⁴ See note 16, *Preliminary Dissertation*.

²⁵ See note 4, *Introduction*.

²⁶ See note 52, *Preliminary Dissertation*.